State constrained stochastic optimal control problems via reachability approach

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General setting

Let $(\Omega, \mathcal{F}, \{\mathbb{F}_t\}_{t\geq 0}, \mathbb{P})$ be a filtered probability space.
System of controlled SDE's in $\mathbb{R}^d$:

$$\begin{cases} \mathrm{d}X(s) = b(s, X(s), u(s))\mathrm{d}s + \sigma(s, X(s), u(s))\mathrm{d}B(s) \\ X(t) = x, \end{cases}$$

where $u \in \mathcal{U} := \{\text{Progr. meas. processes valued in } U\}$ and

- $B(\cdot)$: p-dimensional Brownian motion;
- $U \subset \mathbb{R}^m$: set of control values, compact set;
- $T > 0$: time horizon;
- b and σ: Lipschitz continuous.

$\rightsquigarrow$ $X^u_{t,x}(\cdot)$: unique strong solution associated with the control u.

Optimal control under state constraints

$$\mathcal{K} \subseteq \mathbb{R}^d$$

$$\text{State constraints: } X^u_{t,x}(\cdot) \in \mathcal{K} \text{ a.s.}$$

Main motivations: consider the real world restrictions (physical constraints, presence of obstacles, constraints coming from financial applications, etc.)

$$\upsilon(t,x) = \inf \left\{ \mathbb{E}\left[\psi(X^u_{t,x}(T)) + \int_t^T \ell(s, X^u_{t,x}(s), u(s)) \mathrm{d}s \right] \text{ such that } u \in \mathcal{U} \text{ and } X^u_{t,x}(s) \in \mathcal{K}, \forall s \in [t,T] \text{ a.s.} \right\}$$

- $\psi : \mathbb{R}^d \to \mathbb{R}$ (terminal cost);
- $\ell : [0,T] \times \mathbb{R}^d \times U \to \mathbb{R}$ (running cost).

Example: production planning

Demand:

$$\mathrm{d}X(s) = b(X(s))\mathrm{d}s + \sigma(X(s))\mathrm{d}B(s), \qquad X(t) = x \in \mathbb{R}.$$

Inventory level:

$$\mathrm{d}Y(s) = (u(s) - X(s))\mathrm{d}s, \qquad Y(t) = y.$$

- Production rate (control): $u(\cdot) \in U = [0, u_{\max}]$;
- Capacity (state constraint): $Y(s) \in [0, Y_{\max}] =: \mathcal{K},\ \forall s \in [0, T]$.
- Cost of inventory-production: $\ell(Y(s), u(s)), \psi(Y(T))$.

Optimal control problem:

$$v(t, x, y) := \inf_{u \in \mathcal{U}} \left\{ \mathbb{E}\left[\int_t^T \ell(Y^u_{t,x,y}(s), u(s))\mathrm{d}s + \psi(Y^u_{t,x,y}(T)) \right] : Y^u_{t,x,y}(s) \in [0, Y_{max}] \right\}.$$

Some references

If $\mathcal{K} = \mathbb{R}^d$: v is the unique continuous viscosity solution of second order HJB equation (dynamic programming approach).

If $\mathcal{K} \subset \mathbb{R}^d$: the dynamic programming approach is usually applied under **"controllability conditions"** on the coefficients of the SDE.

- The viability of $\mathcal{K}$ ensures finiteness of v.
  Example. $\mathcal{K}$ closure of a smooth domain, d signed distance to $\partial\mathcal{K}$:
  $$\left.\begin{array}{l} \forall x \in \partial\mathcal{K} : \exists u \in U \text{ s.t. } \sigma(x,u)\cdot \mathbf{n}_{\text{out}}(x) = 0, \\ -b(x,u)\cdot \mathbf{n}_{\text{out}} + \frac{1}{2} Tr[\sigma\sigma^T(x,u) D^2 d] \geq 0 \end{array}\right. \implies \quad \mathcal{K} \text{ is viable.}$$
- HJB characterisation:
  - Soner ('86) et al.: deterministic case;
  - Lasry-Lions ('89): singular boundary conditions;
  - Katsoulakis ('94), Barles-Rouy ('98), Bouchard-Nutz ('12), Ishii-Loreti ('02) et al. : conditions on b and σ.

Aim of this work: characterize v in absence of this kind of assumptions.

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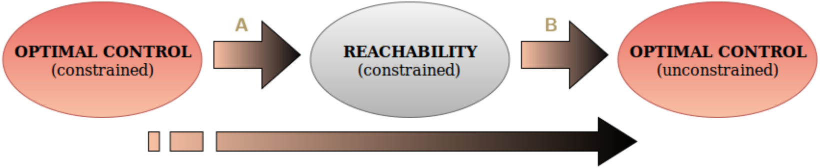


Determ.:	Cardaliaguet-Quincampoix-SaintPierre ('00), Aubin-Frankowska ('96), Altarovici-Bokanowski-Zidani ('13)	Level Set Method (Osher-Sethian ('88)): Falcone-Giorgi-Loreti ('94), Bokanowski-Forcadel-Zidani ('10), Kurzhanski-Varaiya ('06)
Stoch.:	Bouchard-Dang ('12): unconstrained case	Soner-Touzi ('02): unconstr. Bokanowski-AP-Zidani ('14)

Outline

Outline

State constrained VS backward reachability

Consider our **optimal control** problem:

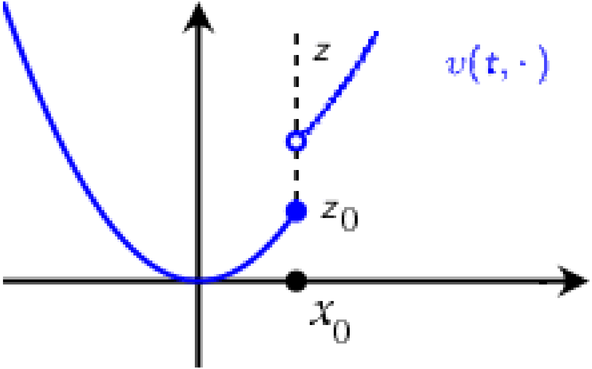
$$v(t,x) = \inf \left\{ \mathbb{E}\left[\psi(X^u_{t,x}(T)) + \int_t^T \ell(s, X^u_{t,x}(s), u(s))\mathrm{d}s \right] : \right.$$
$$\left. u \in \mathcal{U} \text{ and } X^u_{t,x}(s) \in \mathcal{K}, \forall s \in [t,T] \text{ a.s.} \right\}$$

A **backward reachability** problem is defined for a target $\mathcal{T} \in \mathbb{R}^d$ by:

$$\mathcal{R} := \left\{ x \in \mathbb{R}^d : \exists u \in \mathcal{U} \text{ such that } X^u_{t,x}(T) \in \mathcal{T} \text{ a.s.} \right.$$
$$\left. \text{and } X^u_{t,x}(s) \in \mathcal{K}, \forall s \in [t,T] \text{ a.s.} \right\}$$

From optimal control to backward reachability

$$v(t,x) = \inf \Big\{ z \in \mathbb{R} : \exists u \in \mathcal{U} \text{ such that } X^u_{t,x}(s) \in \mathcal{K}\ \forall s \in [t,\,T] \text{ a.s.}$$
$$z \geq \mathbb{E}\Big[\psi(X^u_{t,x}(T)) + \int_t^T \ell(s, X^u_{t,x}(s), u(s))\mathrm{d}s\Big]\Big\} =: \inf\ \mathcal{I}(t,x)$$



From optimal control to backward reachability

Proof.
($\leq$) For any $z \in \mathcal{I}(t, x)$ we have the existence of $u \in \mathcal{U}$ such that

$$X^u_{t,x}(\cdot) \in \mathcal{K} \text{ a.s.} \quad \text{and} \quad z \geq \mathbb{E}\left[\psi(X^u_{t,x}(T)) + \int_t^T \ell(s, X^u_{t,x}(s), u(s))\mathrm{d}s\right].$$

Then, it follows by the very definition of υ that $\upsilon(t, x) \leq z, \forall z \in \mathcal{I}(t, x)$.

($\geq$) For any $u \in \mathcal{U}$ such that $X^u_{t,x}(s) \in \mathcal{K}\ \forall s \in [t, T]$ a.s.

$$\mathbb{E}\left[\psi(X^u_{t,x}(T)) + \int_t^T \ell(s, X^u_{t,x}(s), u(s))\mathrm{d}s\right] \in \mathcal{I}(t, x).$$

From optimal control to backward reachability

Proposition (based on Bouchard-Dang ('12))

Let $\alpha \in L^2$, set of square-integrable $\mathbb{R}^p - \textit{valued}$ predictable processes, and

$$Z^{\alpha}_{t,z}(\cdot) := z + \int_t^{\cdot} \ell(s, X^u_{t,x}(s), u(s))\mathrm{d}s + \int_t^{\cdot} \alpha^T(s)\mathrm{d}B(s).$$

Then

$$\begin{array}{c} \text{Exists } u \in \mathcal{U} : \\ z \geq \mathbb{E}[\psi(X^u_{t,x}(T)) + \int_t^T \ell(s, X^u_{t,x}(s), u(s))\mathrm{d}s] \\ \text{and} \\ X^u_{t,x}(s) \in \mathcal{K}\ \forall s \in [t,T] \text{ a.s.} \end{array} \iff \begin{array}{c} \text{Exist } (u,\alpha) \in \mathcal{U} \times L^2 : \\ Z^{\alpha}_{t,z}(T) \geq \psi(X^u_{t,x}(T)) \text{ a.s.} \\ \text{and} \\ X^u_{t,x}(s) \in \mathcal{K}\ \forall s \in [t,T] \text{ a.s.} \end{array}$$

Elements of the proof. Martingale representation theorem.

The auxiliary backward reachability problem

Let us define the following **"backward reachable set"**:

$$\mathcal{R}_t^{\mathcal{T},\mathcal{K}} := \Big\{ (x,z) \in \mathbb{R}^{d+1} : \exists (u,\alpha) \in \mathcal{U} \times L^2 \text{ such that}$$
$$\Big(Z_{t,x,z}^{\alpha,u}(T) \geq \psi(X_{t,x}^u(T)) \text{ and } X_{t,x}^u(s) \in \mathcal{K} \ \forall s \in [t,T] \Big) \text{ a.s.} \Big\}$$
$$= \Big\{ (x,z) \in \mathbb{R}^{d+1} : \exists (u,\alpha) \in \mathcal{U} \times L^2 \text{ such that}$$
$$\Big((X_{t,x}^u(T), Z_{t,x,z}^{\alpha,u}(T)) \in \mathcal{T} \text{ and } X_{t,x}^u(s) \in \mathcal{K} \ \forall s \in [t,T] \Big) \text{ a.s.} \Big\}$$

where $\mathcal{T} := \text{Epigraph}(\psi)$.

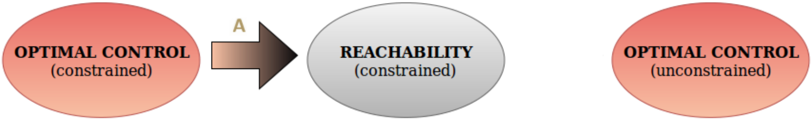
Step A: from optimal control to backward reachability

Thanks to the previous results we get:

$$v(t,x) = \inf \Big\{ z \in \mathbb{R} : \exists u \in \mathcal{U} \text{ such that } X^u_{t,x}(s) \in \mathcal{K} \ \forall s \in [t,T] \text{ a.s.}$$

$$z \geq \mathbb{E}\Big[\psi(X^u_{t,x}(T)) + \int_t^T \ell(s, X^u_{t,x}(s), u(s)) ds \Big] \Big\}$$

$$= \inf \Big\{ z \in \mathbb{R} : (x,z) \in \mathcal{R}^{\mathcal{T},\mathcal{K}}_t \Big\}$$



Outline

The level set approach

$$\mathcal{R}_t^{\mathcal{T},\mathcal{K}} = \Big\{ (x,z) \in \mathbb{R}^{d+1} : \exists (u,\alpha) \in \mathcal{U} \times L^2 \text{ such that } \Big((X_{t,x}^u(T), Z_{t,x,z}^{\alpha,u}(T)) \in \mathcal{T} \text{ and } X_{t,x}^u(s) \in \mathcal{K}\ \forall s \in [t,T] \Big) \text{ a.s.} \Big\}$$

Main idea: interpret $\mathcal{R}_t^{\mathcal{T},\mathcal{K}}$ as a level set of a continuous function that is solution of a suitable PDE.

Advantages:

- Very easy treatment of state constraints;
- Characterization of the whole reachable set;
- Numerical approximation by means of numerical methods for PDEs.

Application of the level set method

Consider the following problem:

$$w(t,x,z) = \inf_{\mathcal{U}\times L^2} \mathbb{E}\left[\Big(\psi(X^u_{t,x}(T)) - Z^{u,\alpha}_{t,z}(T)\Big)_+ + \int_t^T d_{\mathcal{K}}(X^u_{t,x}(s))ds\right]$$

where

$$(\psi(x) - z)_+ := \max(\psi(x) - z, 0) = 0 \Leftrightarrow z \geq \psi(x) \Leftrightarrow (x,z) \in \mathcal{T}$$

and

$$d_{\mathcal{K}}(x) \geq 0 \text{ and } d_{\mathcal{K}}(x) = 0 \Leftrightarrow x \in \mathcal{K}.$$

Assumption: For any (t,x,z) the infimum in w is attained.

Step B: from backward reachability to optimal control

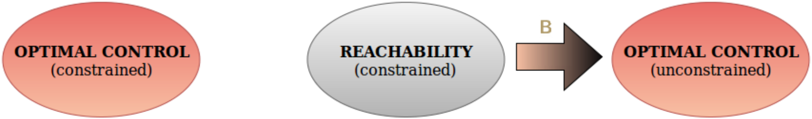
Theorem (Bokanowski-AP-Zidani '16)

One has:

$$\mathcal{R}_t^{\mathcal{T},\mathcal{K}} = \Big\{(x,z) \in \mathbb{R}^{d+1} : w(t,x,z) = 0\Big\}.$$

Therefore:

$$v(t,x) = \inf\Big\{z \in \mathbb{R} : (x,z) \in \mathcal{R}_t^{\mathcal{T},\mathcal{K}}\Big\} = \inf\Big\{z \in \mathbb{R} : w(t,x,z) = 0\Big\}.$$



Step B: from backward reachability to optimal control

Theorem (Bokanowski-AP-Zidani '16)

One has:

$$\mathcal{R}_t^{\mathcal{T},\mathcal{K}} = \Big\{(x,z) \in \mathbb{R}^{d+1} : w(t,x,z) = 0\Big\}.$$

Therefore:

$$\upsilon(t,x) = \inf\Big\{z \in \mathbb{R} : (x,z) \in \mathcal{R}_t^{\mathcal{T},\mathcal{K}}\Big\} = \inf\Big\{z \in \mathbb{R} : w(t,x,z) = 0\Big\}.$$

Conclusions and future projects

- The state constrained optimal control problem has been translated in a state constrained reachability problem in an augmented state and control space (with unbounded controls);
- The level set method has been applied for solving backward reachability problems managing the state constraints by an exact penalization technique;

... moreover ...

- A characterization of the level function w (auxiliary unconstrained OCP with unbounded controls) as unique solution of a **Hamilton-Jacobi-Bellmann equation** has been obtained.

O. Bokanowski, AP, H. Zidani. *State constrained stochastic optimal control problems via reachability approach*. SIAM J. Control and Optim. (2016)

Future projects: reachability in probability

Let $\rho \in [0, 1)$. The aim is to characterize

$$\mathcal{R}_t^{\mathcal{T},\mathcal{K},\rho} := \Big\{ x \in \mathbb{R}^d : \exists u \in \mathcal{U} \text{ such that } \mathbb{P}\Big[X_{t,x}^u(T) \in \mathcal{T} \text{ and } X_{t,x}^u(s) \in \mathcal{K}, \forall s \in [t, T] \Big] > \rho \Big\}$$

One can easily check that defined

$$w(t,x) := \sup_{u \in \mathcal{U}} \mathbb{E}\Big[\mathbb{1}_{\mathcal{K}}(X_{t,x}^u(T) \bigwedge \min_{\theta \in [t,T]} \mathbb{1}_{\mathcal{K}}(X_{t,x}^u(s) \Big],$$

one has

$$\mathcal{R}_t^{\mathcal{T},\mathcal{K},\rho} = \Big\{ x \in \mathbb{R}^d : w(t,x) > \rho \Big\}$$

Problem: Regularization of indicator functions.

M. Asselaou, AP. *A Hamilton-Jacobi-Bellman approach for the numerical computation of probabilistic state constrained reachable sets.* Proceedings NUMOC '17.

Future projects: optimal control with quantile constraints

Let $\rho \in [0,1)$. The aim is to characterize

$$v(t,x) = \inf \Big\{ \mathbb{E}\Big[\psi(X^u_{t,x}(T)) + \int_t^T \ell(s, X^u_{t,x}(s), u(s))\mathrm{d}s\Big] : \\ u \in \mathcal{U} \text{ and } \mathbb{P}\Big[X^u_{t,x}(s) \in \mathcal{K}, \forall s \in [t,T]\Big] > \rho \Big\}.$$

Idea: Use the incator function for rewrite the probability and then apply the martingale representation theorem? (Ongoing with G. Bouveret...).

Thank you very much for your attention!