Stochastic PDEs on networks with non–local boundary conditions and application to finance

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Bibliography

[1] F. C. and L. Di Persio, *Gaussian estimates on networks with dynamic stochastic boundary conditions*, Infinite Dimensional Analysis, Quantum Probability and Related Topics, 20, (2017): 1750001;
[2] F. C. and L. Di Persio, *Stochastic reaction–diffusion equations on networks with dynamic time–delayed boundary conditions*, Journal of Mathematical Analysis and Applications, (2017), 1, 583-603.

Outline

Main motivations

(i) quantum mechanics;

(ii) electrical circuits;

(iii) traffic flow;

(iv) neurobiology;

(v) smart grid optimization;

(vi) system of interconnected banks.

Notation

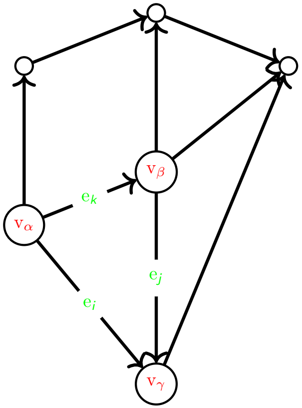
Let us consider a graph $\mathbb{G}$ with:

$n \in \mathbb{N}$ vertices $V = \{\mathrm{v}_1, \ldots, \mathrm{v}_n\}$

$m \in \mathbb{N}$ edges $E = \{\mathrm{e}_1, \ldots, \mathrm{e}_m\}$

Greek letters for vertices $\mathrm{v}_\alpha, \mathrm{v}_\beta, \mathrm{v}_\gamma$

Latin letters for edges $\mathrm{e}_i, \mathrm{e}_j, \mathrm{e}_k$



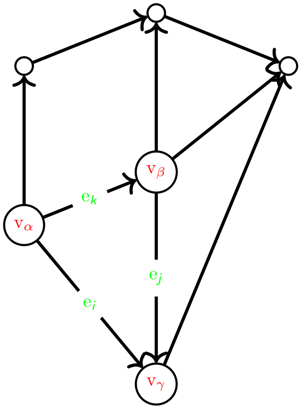
Notation

Incidence matrix

$$\mathcal{I} = (\iota_{ve}) \ , \quad \iota_{ve} = \begin{cases} 1 & \cdot \xrightarrow{e} v \ , \\ -1 & v \xrightarrow{e} \cdot \\ 0 & \text{otherwise} \ ; \end{cases}$$

Adjacency matrix

$$\mathcal{A} = (a_{vw}) \ , \quad a_{vw} = \begin{cases} 1 & v \xrightarrow{e} w \ , \\ 1 & w \xrightarrow{e} v \ , \\ 0 & \text{otherwise} \ ; \end{cases}$$



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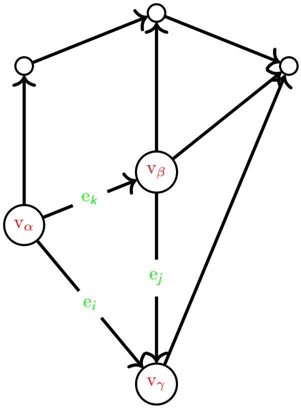
Adjacency matrix

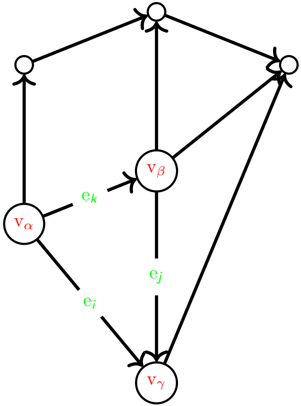
$$\mathcal{A} = (a_{vw}) \; , \quad a_{vw} = \begin{cases} \gamma(e) & v \xrightarrow{\;e\;} w \; , \\ \gamma(e) & w \xrightarrow{\;e\;} v \; , \\ 0 & \text{otherwise} \; ; \end{cases}$$



Consider a diffusion equation on the graph $\mathbb{G}$

$$\dot{u}_j(t,x) = \Delta u_j(t,x)\,,$$

$$u_j(t,x)\,, \quad \text{on the edge} \quad \mathrm{e}_j\,,$$



Main aim

Write the diffusion equation as an abstract operatorial problem

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Boundary conditions?

Continuity in the nodes

$$u_j(t, \mathrm{v}_\alpha) = u_i(t, \mathrm{v}_\alpha) =: d^u_\alpha(t)\,, \quad i,\, j \in \Gamma(\mathrm{v}_\alpha)\,,$$

Continuity in the nodes

$$u_j(t, \mathrm{v}_\alpha) = u_i(t, \mathrm{v}_\alpha) =: d_\alpha^u(t)\,, \quad i,\, j \in \Gamma(\mathrm{v}_\alpha)\,,$$

Kirchhoff condition

$$\sum_{j \in \Gamma(\mathrm{v}_\alpha)} \iota_{\alpha j} u_j'(t, \mathrm{v}_\alpha) = 0\,.$$

Reaction–diffusion equation

$$\begin{cases} \dot{u}_j(t,x) = \sum_{i=1}^{m} \left(c_{ij} u_i'\right)'(t,x) + \sum_{i=1}^{m} p_{ij} u_i(t,x)\,, \\ \\ u_j(0,x) = u_j^0(x)\,, \end{cases}$$

Continuity condition

$$\begin{cases} \dot{u}_j(t,x) = \sum_{i=1}^{m} \left(c_{ij} u_i'\right)'(t,x) + \sum_{i=1}^{m} p_{ij} u_i(t,x)\,, \\ u_j(t,\mathrm{v}_\alpha) = u_l(t,\mathrm{v}_\alpha) =: d_\alpha^u(t)\,, \\ \\ \\ u_j(0,x) = u_j^0(x)\,, \\ \end{cases}$$

Generalized non–local Kirchhoff condition

$$
\begin{cases}
\dot{u}_j(t,x) = \sum_{i=1}^{m} \left(c_{ij} u_i'\right)'(t,x) + \sum_{i=1}^{m} p_{ij} u_i(t,x)\,, \\
u_j(t,\mathrm{v}_\alpha) = u_l(t,\mathrm{v}_\alpha) =: d_\alpha^u(t)\,, \\
\sum_{\beta=1}^{n} b_{\alpha\beta} d_\beta^u(t) = \sum_{i,j=1}^{m} \sum_{\beta=1}^{n} \delta_{\beta j}^{\alpha i} u_j'(t,\mathrm{v}_\alpha)\,, \quad \alpha = n_0+1,\dots,n\,, \\
\\
u_j(0,x) = u_j^0(x)\,,
\end{cases}
$$

Dynamic non–local Kirchhoff condition

$$
\begin{cases}
\dot{u}_j(t,x) = \sum_{i=1}^{m} \left(c_{ij} u_i'\right)'(t,x) + \sum_{i=1}^{m} p_{ij} u_i(t,x)\,, \\
u_j(t,\mathrm{v}_\alpha) = u_l(t,\mathrm{v}_\alpha) =: d_\alpha^u(t)\,, \\
\sum_{\beta=1}^{n} b_{\alpha\beta} d_\beta^u(t) = \sum_{i,j=1}^{m} \sum_{\beta=1}^{n} \delta_{\beta j}^{\alpha i} u_j'(t,\mathrm{v}_\alpha)\,, \quad \alpha = n_0+1,\dots,n\,, \\
\dot{d}_\alpha^u(t) = -\sum_{i,j=1}^{m} \sum_{\beta=1}^{n} \delta_{\beta j}^{\alpha i} u_j'(t,\mathrm{v}_\alpha) + \sum_{\beta=1}^{n} b_{\alpha\beta} d_\beta^u(t)\,, \quad \alpha = 1,\dots,n_0\,, \\
u_j(0,x) = u_j^0(x)\,, \\
d_\alpha^u(0) = d_\alpha^0\,, \quad \alpha = 1,\dots,n_0\,,
\end{cases}
$$

The abstract setting

$$X^2 := \left(L^2([0,1])\right)^m, \quad \mathbb{R}^n,$$

$$\mathcal{X}^2 := X^2 \times \mathbb{R}^n,$$

$$\left\langle \begin{pmatrix} u \\ d^u \end{pmatrix}, \begin{pmatrix} v \\ d^v \end{pmatrix} \right\rangle_{\mathcal{X}^2} := \sum_{j=1}^{m} \int_0^1 u_j(x) v_j(x) dx + \sum_{\alpha=1}^{n} d_\alpha^u d_\alpha^v ,$$

The differential operator

$$
Au = \begin{pmatrix} (c_{1,1}u_1')' + p_{1,1}u_1 & \dots & (c_{1,m}u_1')' + p_{1,m}u_m \\ \vdots & \ddots & \vdots \\ (c_{m,1}u_1')' + p_{m,1}u_1 & \dots & (c_{m,m}u_m')' + p_{m,m}u_m \end{pmatrix},
$$

with domain

$$
D(A) = \Big\{ u \in \left(H^2(0,1)\right)^m \ : \ \exists\, d^u(t) \in \mathbb{R}^n \ \text{s.t.} \ \left(\Phi^+\right)^T d^u(t) = u(0) \ , \\ \left(\Phi^-\right)^T d^u(t) = u(1)\,,\ \Phi_\delta^+ u'(0) - \Phi_\delta^- u'(1) = B_2 d^u(t) \Big\}\ .
$$

Operator matrix

$$\mathcal{A} = \begin{pmatrix} A & 0 \\ C & B_1 \end{pmatrix},$$

with

$$D(\mathcal{A}) = \left\{ \begin{pmatrix} u \\ d^u \end{pmatrix} \in D(A) \times \mathbb{R}^n \,:\, u_i(\mathbf{v}_\alpha) = d^u_\alpha\,, \quad \forall\, i \in \Gamma(\mathbf{v}_\alpha),\, \alpha = 1, \dots, n \right\}.$$

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$C : D(C) := D(A) \to \mathbb{R}^n$ the feedback operator

$$Cu := \left(-\sum_{i,j=1}^{m} \sum_{\beta=1}^{n} \delta^{1i}_{\beta j} u'_j(\mathrm{v}_1), \dots, -\sum_{i,j=1}^{m} \sum_{\beta=1}^{n} \delta^{n_0 i}_{\beta j} u'_j(\mathrm{v}_{n_0}), 0, \dots, 0 \right)^T,$$

The abstract equation

$$\begin{cases} \dot{\mathbf{u}}(t) = \mathcal{A}\,\mathbf{u}(t)\,, \quad t \geq 0\,, \\ \mathbf{u}(0) = \mathbf{u}_0 \in \mathcal{X}^2\,. \end{cases}$$

The abstract equation

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Does $\mathcal{A}$ generate a C_0-semigroup?

Define the sesquilinear form

$$\mathfrak{a}(\mathbf{u}, \mathbf{v}) := \langle Cu', v' \rangle_2 - \langle Pu, v \rangle_2 - \langle B_1 d^u, d^v \rangle_n - \langle B_2 d^u, d^v \rangle_n \, .$$

Define the sesquilinear form

$$\mathfrak{a}(\mathbf{u},\mathbf{v}) := \langle Cu', v' \rangle_2 - \langle Pu, v \rangle_2 - \langle B_1 d^u, d^v \rangle_n - \langle B_2 d^u, d^v \rangle_n .$$

Proposition

The operator associated with the form $\mathfrak{a}$ is the operator $(\mathcal{A}, D(\mathcal{A}))$. Also $(\mathcal{A}, D(\mathcal{A}))$ generates an analytic and compact C_0−semigroup $\mathcal{T}(t)$ on $\mathcal{X}^2$.

Gaussian upper bound

Theorem

The semigroup $\mathcal{T}(t)$, acting on the space $\mathcal{X}^2$ and associated to $\mathfrak{a}$, is ultracontractive, namely there exists a constant $M > 0$ such that

$$\|\mathcal{T}(t)\mathbf{u}\|_{\mathcal{X}^\infty} \leq M t^{-\frac{1}{4}} \|\mathbf{u}\|_{\mathcal{X}^2}\,, \quad t \in [0, T],\, \mathbf{u} \in \mathcal{X}^2\,.$$

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Theorem

The semigroup $\mathcal{T}(t)$ has an integral kernel K_t

$$[\mathcal{T}(t)g](x) = \int_\Omega K_t(x, y) g(y) \mu(dy)\,.$$

It holds the Gaussian upper bound

$$0 \leq K_t(x, y) \leq c_\delta t^{-\frac{1}{2}} e^{-\frac{|x-y|^2}{\sigma t}}\,.$$

Proposition

For any $t \geq 0$*, the semigroup* $\mathcal{T}(t) \in \mathcal{L}_2(\mathcal{X}^2)$*, moreover there exists* $M > 0$ *such that*

$$|\mathcal{T}(t)|_{HS} \leq M t^{-\frac{1}{4}} .$$

The perturbed non–linear stochastic problem

$$
\begin{cases}
\dot{u}_j(t,x) = \sum_{i=1}^m \left(c_{ij} u_i'\right)'(t,x) + \sum_{i=1}^m p_{ij} u_i(t,x) & , \\
u_j(t,\mathrm{v}_\alpha) = u_l(t,\mathrm{v}_\alpha) =: d_\alpha^u(t)\,, \\
\sum_{\beta=1}^n b_{\alpha\beta} d_\beta^u(t) = \sum_{i,j=1}^m \sum_{\beta=1}^n \delta_{\beta j}^{\alpha i} u_j'(t,\mathrm{v}_\alpha)\,, \\
\dot{d}_\alpha^u(t) = -\sum_{i,j=1}^m \sum_{\beta=1}^n \delta_{\beta j}^{\alpha i} u_j'(t,\mathrm{v}_\beta) + \sum_{\beta=1}^n b_{\alpha\beta} d_\beta^u(t) & , \\
u_j(0,x) = u_j^0(x)\,, \\
d_\alpha^u(0) = d_\alpha^0\,,
\end{cases}
$$

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\dot{u}_j(t,x) = \sum_{i=1}^{m} \left(c_{ij} u_i'\right)'(t,x) + \sum_{i=1}^{m} p_{ij} u_i(t,x) + \\
\qquad\qquad + f_j(t,x,u_j(t,x)) & , \\
u_j(t,\mathrm{v}_\alpha) = u_l(t,\mathrm{v}_\alpha) =: d_\alpha^u(t)\,, \\
\sum_{\beta=1}^{n} b_{\alpha\beta} d_\beta^u(t) = \sum_{i,j=1}^{m} \sum_{\beta=1}^{n} \delta_{\beta j}^{\alpha i} u_j'(t,\mathrm{v}_\alpha)\,, \\
\dot{d}_\alpha^u(t) = -\sum_{i,j=1}^{m} \sum_{\beta=1}^{n} \delta_{\beta j}^{\alpha i} u_j'(t,\mathrm{v}_\beta) + \sum_{\beta=1}^{n} b_{\alpha\beta} d_\beta^u(t) & , \\
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\dot{u}_j(t,x) = \sum_{i=1}^{m} \left(c_{ij} u_i'\right)'(t,x) + \sum_{i=1}^{m} p_{ij} u_i(t,x) + \\
\qquad\qquad\qquad + f_j(t,x,u_j(t,x)) + g_j(t,x,u_j(t,x)) \dot{W}_j^1(t,x)\,, \\
u_j(t,\mathrm{v}_\alpha) = u_l(t,\mathrm{v}_\alpha) =: d_\alpha^u(t)\,, \\
\sum_{\beta=1}^{n} b_{\alpha\beta} d_\beta^u(t) = \sum_{i,j=1}^{m} \sum_{\beta=1}^{n} \delta_{\beta j}^{\alpha i} u_j'(t,\mathrm{v}_\alpha)\,, \\
\dot{d}_\alpha^u(t) = -\sum_{i,j=1}^{m} \sum_{\beta=1}^{n} \delta_{\beta j}^{\alpha i} u_j'(t,\mathrm{v}_\beta) + \sum_{\beta=1}^{n} b_{\alpha\beta} d_\beta^u(t) + \tilde{g}_\alpha(t, d_\alpha^u(t)) \dot{W}_\alpha^2(t,\mathrm{v}_\alpha)\,, \\
u_j(0,x) = u_j^0(x)\,, \\
d_\alpha^u(0) = d_\alpha^0\,,
\end{cases}
$$

The abstract equation

$$
\begin{cases} d\mathbf{u}(t) = [\mathcal{A}\mathbf{u}(t) + F(t, \mathbf{u}(t))]\, dt + G(t, \mathbf{u}(t)) dW(t)\,, \quad t \geq 0\,, \\ \mathbf{u}(0) = \mathbf{u}_0 \in \mathcal{X}^2\,, \end{cases}
$$

Theorem

There exists a unique mild solution in the sense that

$$\mathbf{u}(t) = \mathcal{T}(t)\mathbf{u}_0 + \int_0^t \mathcal{T}(t-s)F(s,\mathbf{u}(s))ds + \int_0^t \mathcal{T}(t-s)G(s,\mathbf{u}(s))dW(s)\,.$$

Proof.

□

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Proof.

Main difficulty: treat the stochastic convolution

$$\int_0^t \mathcal{T}(t-s)G(s,\mathbf{u}(s))dW(s)\,.$$

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Proof.

Main difficulty: treat the stochastic convolution

$$\int_0^t \mathcal{T}(t-s)G(s,\mathbf{u}(s))dW(s)\,.$$

Standard assumption $G(s,\mathbf{u}(s)) \in \mathcal{L}_2(\mathcal{X}^2)$

□

Theorem

There exists a unique mild solution in the sense that

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Proof.

Main difficulty to treat the stochastic convolution

$$\int_0^t \mathcal{T}(t-s)G(s,\mathbf{u}(s))dW(s)\,.$$

We only require $G(s,\mathbf{u}(s)) \in \mathcal{L}(\mathcal{X}^2)$

□

proof continued...

Recall propositions above:

$\mathcal{A}$ generates an analytic C_0-semigroup and

$$|\mathcal{T}(t)|_{HS} \leq Mt^{-\frac{1}{4}} .$$

proof continued...

Recall propositions above:

$\mathcal{A}$ generates an analytic C_0-semigroup and

$$|\mathcal{T}(t)|_{HS} \leq Mt^{-\frac{1}{4}}\,.$$

$$\mathcal{T}(t)G(t,\mathbf{u}(t)) \in \mathcal{L}_2(\mathcal{X}^2)\,:$$

$$|\mathcal{T}(t)G(t,\mathbf{u}(t))|_{\mathcal{L}_2(\mathcal{X}^2)} \leq Ct^{-\frac{1}{4}}(1+|\mathbf{u}|)\,.$$

Reaction–diffusion equation

$$
\begin{cases}
\dot{u}_j(t,x) = \left(c_j u_j'\right)'(t,x)\,, \\
\\
\\
u_j(0,x) = u_j^0(x)\,, \\
\\
\\
\end{cases}
$$

Continuity condition

$$\begin{cases} \dot{u}_j(t,x) = \left(c_j u_j'\right)'(t,x)\,, \\ u_j(t,\mathrm{v}_\alpha) = u_l(t,\mathrm{v}_\alpha) =: d^\alpha(t)\,, \\ \\ u_j(0,x) = u_j^0(x)\,, \\ \\ \end{cases}$$

Dynamic time–delayed Kirchhoff condition

$$
\begin{cases}
\dot{u}_j(t,x) = \left(c_j u_j'\right)'(t,x)\,, \\
u_j(t,\mathrm{v}_\alpha) = u_l(t,\mathrm{v}_\alpha) =: d^\alpha(t)\,, \\
\dot{d}^\alpha(t) = -\sum_{j=1}^m \phi_{j\alpha} u_j'(t,\mathrm{v}_\alpha) + b_\alpha d^\alpha(t) + \int_{-r}^0 d^\alpha(t+\theta)\mu(d\theta)\,, \\
u_j(0,x) = u_j^0(x)\,, \\
d^\alpha(0) = d_\alpha^0\,, \\
\end{cases}
$$

Dynamic time–delayed Kirchhoff condition

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\dot{d}^\alpha(t) = -\sum_{j=1}^m \phi_{j\alpha} u_j'(t,\mathrm{v}_\alpha) + b_\alpha d^\alpha(t) + \int_{-r}^0 d^\alpha(t+\theta)\mu(d\theta)\,, \\
u_j(0,x) = u_j^0(x)\,, \\
d^\alpha(0) = d_\alpha^0\,, \\
d^\alpha(\theta) = \eta_\alpha^0(\theta)\,.
\end{cases}
$$

The abstract setting

$$X^2 := \left(L^2([0,1])\right)^m , \quad Z^2 := L^2([-r,0];\mathbb{R}^n) ,$$
$$\mathcal{X}^2 := X^2 \times \mathbb{R}^n , \quad \mathcal{E}^2 := \mathcal{X}^2 \times Z^2 ,$$

The abstract setting

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$$\mathcal{X}^2 := X^2 \times \mathbb{R}^n , \quad \mathcal{E}^2 := \mathcal{X}^2 \times Z^2 ,$$

Consider the process $d : [-r, T] \to \mathbb{R}^n$ and define the segment

$$d_t : [-r,0] \to \mathbb{R}^n , \quad [-r,0] \ni \theta \mapsto d_t(\theta) := d(t+\theta) \in \mathbb{R}^n .$$

The abstract PDE

$$\begin{cases} \dot{u}(t) = A_m u(t)\,, \quad t \in [0, T]\,, \\ \dot{d}(t) = Cu(t) + \Phi d_t + \tilde{B} d(t)\,, \quad t \in [0, T]\,, \\ \dot{d}_t = A_\theta d_t\,, \quad t \in [0, T]\,, \\ Lu(t) = d(t)\,, \\ u(0) = u_0 \in X^2\,, \quad d_0 = \eta \in Z^2\,, \quad d(0) = d^0 \in \mathbb{R}^n\,, \end{cases}$$

The abstract PDE

$$A_m u(t,x) = \begin{pmatrix} \frac{\partial}{\partial x}\left(c_j(x)\frac{\partial}{\partial x}u_1(t,x)\right) & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \frac{\partial}{\partial x}\left(c_m(x)\frac{\partial}{\partial x}u_m(t,x)\right) \end{pmatrix},$$

and such that $A_m : D(A_m) \subset X^2 \to X^2$, with domain

$$D(A) := \left\{ u \in \left(H^2([0,1])\right)^m \,:\, \exists d \in \mathbb{R}^n \,:\, Lu = d \right\},$$

The abstract PDE

$L : \left(H^1([0,1])\right)^m \to \mathbb{R}^n$ is the boundary evaluation operator

$$Lu(t,x) := \left(d^1(t), \dots, d^n(t)\right)^T \,, \quad d^\alpha(t) := u_j(t, \mathrm{v}_\alpha)\,.$$

The abstract PDE

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$$Lu(t,x) := \left(d^1(t), \ldots, d^n(t)\right)^T \,, \quad d^\alpha(t) := u_j(t, \mathrm{v}_\alpha)\,.$$

$C : D(A) \to \mathbb{R}^n$ is the feedback operator

$$Cu(t,x) := \left(-\sum_{j=1}^m \phi_{j1} u_j'(t, \mathrm{v}_1), \ldots, -\sum_{j=1}^m \phi_{jn} u_j'(t, \mathrm{v}_n)\right)^T \,.$$

The abstract PDE

$\Phi : H^1([-r,0];\mathbb{R}^n) \to \mathbb{R}^n$ is the delay operator

$$\Phi d_t = \int_{-r}^{0} d^\alpha(t+\theta)\mu(d\theta)\,.$$

The abstract PDE

$\Phi : H^1([-r,0];\mathbb{R}^n) \to \mathbb{R}^n$ is the delay operator

$$\Phi d_t = \int_{-r}^{0} d^\alpha(t+\theta)\mu(d\theta)\,.$$

$A_\theta : D(A_\theta) \subset Z^2 \to Z^2$

$$A_\theta \eta := \frac{\partial}{\partial\theta}\eta(\theta)\,, \quad D(A_\theta) = \{\eta \in H^1([-r,0];\mathbb{R}^n)\ :\ \eta(0) = d^0\}\,,$$

The abstract equation

$$\begin{cases} \dot{\mathbf{u}}(t) = \mathcal{A}\mathbf{u}(t)\,, \quad t \in [0, T]\,, \\ \mathbf{u}(0) = \mathbf{u}_0 \in \mathcal{E}^2\,, \end{cases}$$

$\mathcal{A}$ is defined as

$$\mathcal{A} := \begin{pmatrix} A_m & 0 & 0 \\ C & B & \Phi \\ 0 & 0 & A_\theta \end{pmatrix} ,$$

with domain $D(\mathcal{A}) := D(A_m) \times D(A_\theta)$.

The abstract equation

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Does $\mathcal{A}$ generate a C_0-semigroup?

On the infinitesimal generator

$$\mathcal{A} := \begin{pmatrix} A_m & 0 & 0 \\ C & B & \Phi \\ 0 & 0 & A_\theta \end{pmatrix} ,$$

On the infinitesimal generator

$$\mathcal{A}_0 := \begin{pmatrix} A_m & 0 & 0 \\ C & B & 0 \\ 0 & 0 & A_\theta \end{pmatrix},$$

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$$\mathcal{A}_0 := \begin{pmatrix} A_m & 0 & 0 \\ C & B & 0 \\ 0 & 0 & A_\theta \end{pmatrix},$$

$$A_\mathfrak{a} := \begin{pmatrix} A_m & 0 \\ C & B \end{pmatrix},$$

On the infinitesimal generator

$$\mathcal{A}_0 := \begin{pmatrix} A_m & 0 & 0 \\ C & B & 0 \\ 0 & 0 & A_\theta \end{pmatrix},$$

$$A_{\mathfrak{a}} := \begin{pmatrix} A_m & 0 \\ C & B \end{pmatrix},$$

$$\mathcal{A}_0 := \begin{pmatrix} A_{\mathfrak{a}} & 0 \\ 0 & A_\theta \end{pmatrix}, \quad D(\mathcal{A}_0) = D(\mathcal{A}),$$

Theorem

The matrix operator $(\mathcal{A}_0, D(\mathcal{A}_0))$ *generates a* C_0-*semigroup given by*

$$\mathcal{T}_0(t) = \left(\begin{array}{c|c} \begin{matrix} \Large T_{\mathfrak{a}}(t) & \\ & \end{matrix} & \begin{matrix} 0 \\ 0 \end{matrix} \\ \hline \begin{matrix} 0 & \qquad T_t \end{matrix} & T_0(t) \end{array}\right),$$

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$T_{\mathfrak{a}}$ *is the C_0-semigroup generated by $(A_{\mathfrak{a}}, D(A_{\mathfrak{a}}))$*

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$T_{\mathfrak{a}}$ is the C_0-semigroup generated by $(A_{\mathfrak{a}}, D(A_{\mathfrak{a}}))$
$T_0(t)$ is the nilpotent left-shift semigroup

$$\left(T_0(t)\eta\right)(\theta) := \begin{cases} \eta(t+\theta) & t+\theta \leq 0\,, \\ 0 & t+\theta > 0\,, \end{cases}, \quad \eta \in Z^2\,,$$

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$T_t : \mathbb{R}^n \to Z^2$ *is defined by*

$$(T_t d)(\theta) := \begin{cases} e^{(t+\theta)B} d & -t < \theta \le 0\,, \\ 0 & -r \le \theta \le -t\,, \end{cases}, \quad d \in \mathbb{R}^n\,,$$

$e^{(t+\theta)B}$ *being the semigroup generated by the finite dimensional* $n \times n$ *matrix* B.

The Miyadera-Voigt perturbation theorem

Theorem

Let $(G, D(G))$ be the generator of a C_0 semigroup $(S(t))_{t\geq 0}$. Assume that there exist constants $t_0 > 0$ and $0 \leq q \leq 1$, such that

$$\int_0^{t_0} \|KS(t)x\| dt \leq q\|x\|\,, \quad \forall\, x \in D(G)\,.$$

Then $(G + K, D(G))$ generates a strongly continuous semigroup $(U(t))_{t\geq 0}$ on X, which satisfies

$$U(t)x = S(t)x + \int_0^t S(t-s)KU(s)xds\,,$$

and

$$\int_0^{t_0} \|KU(t)x\| dt \leq \frac{q}{1-q}\|x\|\,, \quad \forall\, x \in D(G)\,,\ t \geq 0\,.$$

$$\mathcal{A}_1 := \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \Phi \\ 0 & 0 & 0 \end{pmatrix},$$

$$\mathcal{A} = \mathcal{A}_0 + \mathcal{A}_1 \, .$$

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$$\mathcal{A} = \mathcal{A}_0 + \mathcal{A}_1\,.$$

Theorem

The operator $(\mathcal{A}, D(\mathcal{A}))$ *generates a strongly continuous semigroup.*

Proof.

Apply Miyadera-Voigt perturbation theorem. □

The perturbed non–linear stochastic problem

$$
\begin{cases}
\dot{u}_j(t,x) = \left(c_j u_j'\right)'(t,x) \\
\\
u_j(t,\mathrm{v}_\alpha) = u_l(t,\mathrm{v}_\alpha) =: d^\alpha(t)\,, \\
\dot{d}^\alpha(t) = -\sum_{j=1}^m \phi_{j\alpha} u_j'(t,\mathrm{v}_\alpha) + b_\alpha d^\alpha(t) + \int_{-r}^0 d^\alpha(t+\theta)\mu(d\theta) \\
\\
u_j(0,x) = u_j^0(x)\,, \\
d^\alpha(0) = d_\alpha^0\,, \\
d^\alpha(\theta) = \eta_\alpha^0(\theta)\,.
\end{cases}
$$

The perturbed non–linear stochastic problem

$$
\begin{cases}
\dot{u}_j(t,x) = \left(c_j u_j'\right)'(t,x) + f_j(t,x,u_j(t,x)) \\
\\
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\end{cases}
$$

The perturbed non–linear stochastic problem

$$\begin{cases} \dot{u}_j(t,x) = \left(c_j u_j'\right)'(t,x) + f_j(t,x,u_j(t,x)) + \\ \qquad\qquad\qquad\qquad + g_j(t,x,u_j(t,x))\dot{W}_j^1(t,x)\,, \\ u_j(t,\mathrm{v}_\alpha) = u_l(t,\mathrm{v}_\alpha) =: d^\alpha(t)\,, \\ \dot{d}^\alpha(t) = -\sum_{j=1}^m \phi_{j\alpha} u_j'(t,\mathrm{v}_\alpha) + b_\alpha d^\alpha(t) + \int_{-r}^0 d^\alpha(t+\theta)\mu(d\theta) \\ \qquad\qquad\qquad\qquad + \tilde{g}_\alpha(t,d^\alpha(t),d_t^\alpha)\dot{W}_\alpha^2(t,\mathrm{v}_\alpha)\,, \\ u_j(0,x) = u_j^0(x)\,, \\ d^\alpha(0) = d_\alpha^0\,, \\ d^\alpha(\theta) = \eta_\alpha^0(\theta)\,. \end{cases}$$

Assumption

$$|g_j(t,x,y_1)| \le C_j\,, \quad |g_j(t,x,y_1) - g_j(t,x,y_2)| \le K_j|y_1 - y_2|\,;$$

$$|\tilde{g}_\alpha(t,x,\eta)| \le C_\alpha\,, \quad |\tilde{g}_\alpha(t,x,\eta) - \tilde{g}_\alpha(t,y,\zeta)| \le K_\alpha(|x-y|_n + |\eta-\zeta|_{Z^2})\,.$$

$$|f_j(t,x,y_1)| \le C_j\,, \quad |f_j(t,x,y_1) - f_j(t,x,y_2)| \le K_j|y_1 - y_2|\,.$$

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Remark

$f_j(t,x,y)$ can be assumed also to be non–Lipschitz of polynomial growth.

The abstract equation

$$\begin{cases} d\mathbf{X}(t) = [\mathcal{A}\mathbf{X}(t) + F(t,\mathbf{X})]\,dt + G(t,\mathbf{X}(t))dW(t)\,, & t \geq 0\,, \\ \mathbf{X}(0) = \mathbf{X}_0 \in \mathcal{E}^2\,, \end{cases}$$

Theorem

There exists a unique mild solution in the sense that

$$\mathbf{X}(t) = \mathcal{T}(t)\mathbf{X}_0 + \int_0^t \mathcal{T}(t-s)F(s,\mathbf{X}(s))ds + \int_0^t \mathcal{T}(t-s)G(s,\mathbf{X}(s))dW(s)\,.$$

Proof.

□

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Main difficulty: treat the stochastic convolution

$$\int_0^t \mathcal{T}(t-s)G(s,\mathbf{u}(s))dW(s)\,.$$

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Standard assumption $G(s,\mathbf{u}(s)) \in \mathcal{L}_2(\mathcal{X}^2)$

□

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There exists a unique mild solution in the sense that

$$\mathbf{X}(t) = \mathcal{T}(t)\mathbf{X}_0 + \int_0^t \mathcal{T}(t-s)F(s,\mathbf{X}(s))ds + \int_0^t \mathcal{T}(t-s)G(s,\mathbf{X}(s))dW(s)\,.$$

Proof.

Main difficulty to treat the stochastic convolution

$$\int_0^t \mathcal{T}(t-s)G(s,\mathbf{u}(s))dW(s)\,.$$

We only require $G(s,\mathbf{u}(s)) \in \mathcal{L}(\mathcal{X}^2)$

□

proof continued...

The matrix operator $\mathcal{A}$ contains $A_\theta := \frac{\partial}{\partial\theta}$

$\mathcal{A}$ does not generate an analytic C_0-semigroup

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The matrix operator $\mathcal{A}$ contains $A_\theta := \frac{\partial}{\partial \theta}$

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Proposition

$\mathcal{T}(t)G(s, \mathbf{X}) \in \mathcal{L}_2(\mathcal{X}^2; \mathcal{E}^2)$ *such that*

$$|\mathcal{T}(t)G(s, \mathbf{X})|_{HS} \leq Mt^{-\frac{1}{4}}(1 + |\mathbf{X}|_{\mathcal{E}^2})$$

Proof.

Technical computations exploiting the explicit form for $\mathcal{T}(t)$. □

Application to optimal control

$$\begin{cases} d\mathbf{X}^z(t) = [\mathcal{A}\mathbf{X}^z(t) + F(t,\mathbf{X}^z) + G(t,\mathbf{X}^z(t))R(t,\mathbf{X}^z(t),z(t))]\,dt + \\ \qquad\qquad\qquad\qquad\qquad + G(t,\mathbf{X}^z(t))dW(t)\,, \\ \mathbf{X}^z(t_0) = \mathbf{X}_0 \in \mathcal{E}^2\,, \end{cases}$$

$$J(t_0,\mathbf{X}_0,z) = \mathbb{E}\int_{t_0}^{T} l\,(t,\mathbf{X}^z(t),z(t))\,dt + \mathbb{E}\varphi(\mathbf{X}^z(T)) \to \min\,.$$

Admissible Control System (acs)

$$\left(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P}, (W(t))_{t\geq 0}, z\right)$$

- $\left(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P}\right)$ is a complete probability space, where the filtration $(\mathcal{F}_t)_{t\geq 0}$ satisfies the usual assumptions;
- $(W(t))_{t\geq 0}$ is a $\mathcal{F}_t-$adapted Wiener process taking values in $\mathcal{E}^2$;
- z is a process taking values in the space Z, predictable with respect to the filtration $(\mathcal{F}_t)_{t\geq 0}$, and such that $z(t) \in \mathcal{Z}$ $\mathbb{P}-$a.s., for almost any $t \in [t_0, T]$, being $\mathcal{Z}$ a suitable domain of Z.

Assumption

$$
\begin{aligned}
&|R(t,\mathbf{X},z) - R(t,\mathbf{X},z)|_{\mathcal{E}^2} \le C_R(1 + |\mathbf{X}|_{\mathcal{E}^2} + |\mathbf{Y}|_{\mathcal{E}^2})^m |\mathbf{X} - \mathbf{Y}|_{\mathcal{E}^2} \,, \\
&|R(t,\mathbf{X},z)|_{\mathcal{E}^2} \le C_R \,;
\end{aligned}
$$

$$
\begin{aligned}
&|l(t,\mathbf{X},z) - l(t,\mathbf{X},z)| \le C_l(1 + |\mathbf{X}|_{\mathcal{E}^2} + |\mathbf{Y}|_{\mathcal{E}^2})^m |\mathbf{X} - \mathbf{Y}|_{\mathcal{E}^2} \,, \\
&|l(t,0,z)|_{\mathcal{E}^2} \ge -C \,, \\
&\inf_{z \in \mathcal{Z}} l(t,0,z) \le C_l \,;
\end{aligned}
$$

$$
|\varphi(\mathbf{X}) - \varphi(\mathbf{Y})| \le C_\varphi(1 + |\mathbf{X}|_{\mathcal{E}^2} + |\mathbf{Y}|_{\mathcal{E}^2})^m |\mathbf{X} - \mathbf{Y}|_{\mathcal{E}^2} \,.
$$

Girsanov theorem

$$W^{\zeta}(t) := W(t) - \int_{t_0 \wedge t}^{t \wedge T} R(s, \mathbf{X}(s), \zeta) ds \,,$$

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The HJB equation

$$\psi(t,\mathbf{X},\mathbf{Y}) := -\inf_{z\in\mathcal{Z}}\{l(t,\mathbf{X},z)+\mathbf{Y}R(t,\mathbf{X},z)\}\,,$$

$$\Gamma(t,\mathbf{X},\mathbf{Y}) := \{z\in\mathcal{Z}\,:\,\psi(t,\mathbf{X},\mathbf{Y})+l(t,\mathbf{X},z)+\mathbf{v}R(t,\mathbf{X},z)=0\}\,,$$

$$\begin{cases}\frac{\partial w(t,\mathbf{X})}{\partial t}+\mathcal{L}_t w(t,\mathbf{X}) = \psi(t,\mathbf{X},\nabla^G w(t,\mathbf{X}))\,,\\ w(T,\mathbf{X})=\varphi(\mathbf{X})\,,\end{cases}$$

∇^G being the generalized directional gradient.

Theorem

Let w be a mild solution to the HJB equation, and chose ρ to be an element of the generalized directional gradient $\nabla^G w$. Then, for all ACS, we have that $J(t_0, \mathbf{X}_0, z) \geq w(t_0, \mathbf{X}_0)$, and the equality holds if and only if the following feedback law is verified by z and $\mathbf{u}^z$

$$z(t) = \Gamma\left(t, \mathbf{X}^z(t), G(t, \rho(t, \mathbf{X}^z(t))\right), \quad \mathbb{P}- \ a.s. \ \ for\ a.a. \ \ t \in [t_0, T].$$

Moreover, if there exists a measurable function $\gamma : [0, T] \times \mathcal{E}^2 \times \mathcal{E}^2 \to \mathcal{Z}$ with

$$\gamma(t, \mathbf{X}, \mathbf{Y}) \in \Gamma(t, \mathbf{X}, \mathbf{Y}), \quad t \in [0, T], \mathbf{X}, \mathbf{Y} \in \mathcal{X}^2,$$

then there also exists, at least one ACS such that

$$\bar{z}(t))\gamma(t, \mathbf{X}^z(t), \rho(t, \mathbf{X}^z(t))), \quad \mathbb{P}- \ a.s. \ \ for\ a.a. \ \ t \in [t_0, T].$$

Eventually, we have that $\mathbf{X}^{\bar{z}}$ is a mild solution.

System of interconnected banks

Works in progress with L. Di Persio (UniVr), L. Prezioso (UniVr-UniTn), A. Bressan (Penn State University)and Y. Jiang (Penn State University).

System of interconnected banks

Works in progress with L. Di Persio (UniVr), L. Prezioso (UniVr-UniTn), A. Bressan (Penn State University)and Y. Jiang (Penn State University).

- Multiple defualts of banks;
- Optimal control with terminal probability constraints;
- Stackelberg equilibrium;
- Stochastic impulse control.

System interconnected banks: the setting

- value of the $i-$th bank associated to the vertex v_i, $i = 1, \ldots, n$;
- liabilities matrix $\mathcal{L}(t) = \left(L_{i,j}(t)\right)_{n\times n}$
- $u_i(t)$ the payment made at time $t \in [0, T]$ by v_i;
- $\bar{u}_i(t) = \sum_{j=1}^{n} L_{i,j}(t)$ the total nominal obligation of the node i towards all other nodes;
- relative liabilities matrix $\Pi(t) = (\pi_{i,j}(t))$ defined as

$$\pi_{i,j}(t) = \begin{cases} \frac{L_{i,j}(t)}{\bar{u}_i(t)} & \bar{u}_i(t) > 0\,, \\ 0 & \text{otherwise} \end{cases}.$$

- the cash inflow of the node i is given by $\sum_{j=1}^{n} \left(\Pi_{i,j}(t)\right)^T u_j(t)$.

System interconnected banks: the setting

total value of node v_i at time $t \in [0, T]$

$$\bar{V}^i(t) = \sum_{j=1}^{n} \left(\Pi_{i,j}(t)\right)^T u_j(t) + X^i(t) - u_i(t)\,.$$

System interconnected banks: the setting

- liabilities evolve according to

$$\frac{d}{dt} L_{i,j}(t) = \mu_{ij} L_{i,j}(t) \,,$$

- exogenous asset $X^i(t)$ evolves according to

$$dX^i(t) = X^i(t) \left(\mu^i dt + \sigma^i dW^i(t) \right) \,, \quad i = 1, \dots, n \,.$$

- continuous (deterministic) default boundaries for bank i

$$X^i(t) \leq v^i(t) := \begin{cases} R^i \left(\bar{u}_i(t) - \sum_{j=1}^n \left(\Pi_{i,j}(t) \right)^T \bar{u}_j(t) \right) & t < T \,, \\ \bar{u}_i(t) - \sum_{j=1}^n \left(\Pi_{i,j}(t) \right)^T \bar{u}_j(t) & t = T \,, \end{cases}$$

- R^i, $i = 1, \dots, n$, representing the recovery rate of the bank i.

System interconnected banks: the optimal control problem

- financial supervisor, (*lender of last resort*, (LOLR)), aims at saving the network from default;
- LOLR minimizes the quadratic cost whilst maximizing the distance of each bank from the respective default boundary

$$J(x,\alpha) := \mathbb{E}\left[\int_0^{\hat{\tau}} \left(-\mathbf{L}\left(\mathbf{X}(t)\right) + \frac{1}{2}\|\alpha(t)\|^2\right) dt - \mathbf{G}\left(\mathbf{X}(\hat{\tau}^n)\right)\right] ,$$

- $\hat{\tau}$ random terminal time of default;
- controlled process

$$dX^i(t) = X^i(t)\left(\mu^i dt + \sigma^i dW^i(t)\right) + \alpha^i(t)dt\,, \quad i = 1, \dots, n\,.$$

System interconnected banks: the optimal control problem

- after first default wehave a new system $i = 1, \dots, n$

$$dX_1^i(t) = X_1^i(t)\left(\mu_1^i dt + \sigma_1^i dW^i(t)\right) + \alpha_1^i(t)dt\,, \quad i = 1, \dots, n-1\,.$$

- LOLR minimizes the quadratic cost whilst maximizing the distance of each bank from the respective default boundary

$$J_1(x,\alpha) := \mathbb{E}\left[\int_{\hat\tau}^{\hat\tau^1}\left(-\mathbf{L}_1\left(\mathbf{X}_1(t)\right) + \frac{1}{2}\|\alpha_1(t)\|^2\right)dt - \mathbf{G}_1\left(\mathbf{X}_1(\hat\tau^1)\right)\right]\,,$$

- $\hat\tau^1$ random terminal time of default;

System interconnected banks: the optimal control problem

- after first default wehave a new system $i = 1, \dots, n$

$$dX_1^i(t) = X_1^i(t)\left(\mu_1^i dt + \sigma_1^i dW^i(t)\right) + \alpha_1^i(t)dt\,, \quad i = 1, \dots, n-1\,.$$

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- $\hat{\tau}^1$ random terminal time of default;
- and so on until no nodes are left in the system;

System interconnected banks: the optimal control problem

- multiple optimal control problems with random terminal time;
- stochastic maximum principle for global multiple stochastic optimal control problem;

The maximum principle

Theorem

[Maximum Principle]

$$\partial_a H(t, \bar{\mathbf{X}}(t), \bar{\alpha}(t), \bar{Y}(t), \bar{Z}(t))\,(\bar{\alpha}(t) - \tilde{\alpha}) \leq 0\,,$$

where each $(Y^{\pi^k}(t), Z^{\pi^k}(t))$ *solves the following BSDE's*

$$\begin{cases} -dY^{\pi^{n-1}}(t) & = \partial_x H^{\pi^{n-1}}(t, \mathbf{X}^{\pi^{n-1}}(t), \alpha^{\pi^{n-1}}(t), Y^{\pi^{n-1}}(t), Z^{\pi^{n-1}}(t))dt - Z^{\pi^{n-1}} dW(t)\,, \\ Y^{\pi^{n-1}}(\hat{\tau}^n) & = \partial_x \mathbf{G}^{\pi^{n-1}}(\hat{\tau}^n, \mathbf{X}^{\pi^{n-1}}(\hat{\tau}^n))\,, \end{cases}$$

$$\begin{cases} -dY^{\pi^k}(t) & = \partial_x H^{\pi^k}(t, \mathbf{X}^{\pi^k}(t), \alpha^{\pi^k}(t), Y^{\pi^k}(t), Z^{\pi^k}(t))dt - Z^{\pi^k} dW(t)\,, \\ Y^{\pi^k}(\hat{\tau}^{k+1}) & = \partial_x \mathbf{G}^{\pi^k}(\hat{\tau}^{k+1}, \mathbf{X}^{\pi^k}(\hat{\tau}^{k+1})) + \bar{Y}^{k+1}(\hat{\tau}^{k+1})\,, \end{cases}$$

$$\begin{cases} -dY^0(t) = \partial_x H^0(t, \mathbf{X}^0(t), \alpha^0(t), Y^0(t), Z^0(t))dt - Z^0 dW(t)\,, \\ Y^0(\hat{\tau}) = \partial_x \mathbf{G}^0(\tau_1, \mathbf{X}^0(\tau_1)) + \bar{Y}^1(\tau_1)\,, \end{cases}$$

The optimal control with constrained probability of success

- LOLR minimizes amount of money lent

$$J(x,\alpha) = \frac{1}{2}\int_0^T \|\alpha(s)\|^2 ds\,;$$

under fixed probability of default

$$\mathbb{P}\left(X^i(T) \geq v^i(T)\right) \geq q^i\,, \quad i = 1,\ldots,n\,,$$

- controlled process

$$dX^i(t) = X^i(t)\left(\mu^i dt + \sigma^i dW^i(t)\right) + \alpha^i(t)dt\,, \quad i = 1,\ldots,n\,.$$

The optimal control with constrained probability of success

- two regions for the optimal solution:

The optimal control with constrained probability of success

- two regions for the optimal solution:
- Region I: the probability constraints is satisfied;
- optimal solution $\alpha(t) \equiv 0$;

The optimal control with constrained probability of success

- two regions for the optimal solution:
- Region I: the probability constraints is satisfied;
- optimal solution $\alpha(t) \equiv 0$;
- Region II: the probability constraints is not satisfied;
- we guess $\alpha^i(t) = \psi(t) X^i(t)$
- optimal solution

$$\psi^i = \frac{\ln v^i(T) - \ln x_0}{t_1} - \left(\sqrt{2} Erf^{-1}\left(1 - 2q^i\right)\right) \sigma^i \frac{1}{\sqrt{T}} + \frac{(\sigma)^2}{2} - \mu^i \,.$$

Thank you for your attention!