Hitting probabilities for systems of stochastic partial differential equations

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Based on joint works with:

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Overview

- Introduction to the problems of hitting probabilities and of polarity of points
- Existing results for Gaussian random fields
- Results and methods for non-linear systems of s.p.d.e.'s (mostly in non-critical dimensions)
- Results and methods for critical dimensions

Hitting probabilities and polarity of points for random fields

Let $U = (U(x),\ x \in \mathbb{R}^k)$ be an $\mathbb{R}^d$-valued continuous stochastic process.

Fix $I \subset \mathbb{R}^k$, compact with positive Lebesgue measure.

The range of U over I is the random compact set

$$U(I) = \{U(x),\ x \in I\}.$$

Question 1. (Hitting probabilities) For $A \subset \mathbb{R}^d$, what are bounds on

$$P\{U(I) \cap A \neq \emptyset\}\,?$$

Question 2. (Polarity of points) Fix $z \in \mathbb{R}^d$. Does U fail to hit z, that is,

$$P\{\exists x \in I : U(x) = z\} = 0\,?$$

Polarity. If $P\{\exists x \in I : U(x) = z\} = 0$, then z is polar for U.

Typically, there is a critical dimension Q such that:

- if $d < Q$, then points are not polar.
- if $d > Q$, then points are polar.
- at the critical valued $d = Q$, ???

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First example: Brownian motion

Theorem 1 (Paul Lévy)

Let $B = (B(t),\ t \in \mathbb{R}_+)$ be an $\mathbb{R}^d$-valued Brownian motion. Then points are polar for B if and only if $d \geqslant 2$.

Remark

(a) The critical dimension is $Q = 2$, and points are polar in dimension $d = 2$.

(b) In dimension $d = 1$, it is clear that points are not polar (Intermediate Value Theorem).

(c) In dimensions $d \geqslant 3$, it is fairly straightforward to check that points are polar (explanation later).

(d) In the critical dimension $d = 2$, Lévy gave a clever argument that uses specific properties of Brownian motion (Markov property, time-reversal, etc.).

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Second example: the Brownian sheet

Let $(W(x),\ x \in \mathbb{R}_+^k)$ denote an k-parameter $\mathbb{R}^d$-valued Brownian sheet, that is, a centered continuous Gaussian random field

$$W(x) = (W_1(x), \dots, W_d(x))$$

with covariance

$$E[W_i(x)W_j(y)] = \delta_{i,j} \prod_{\ell=1}^{k} \min(x_\ell, y_\ell), \qquad i,j \in \{1, \dots, d\},$$

where $x = (x_1, \dots, x_k)$ and $y = (y_1, \dots, y_k)$.

Theorem 2 (Khoshnevisan and Shi, 1999)

Fix $M > 0$. Let I be a box. There exists $0 < C < \infty$ such that for all compact sets $A \subset B(0, M)$ $(\subset \mathbb{R}^d)$,

$$\frac{1}{C}\ \mathrm{Cap}_{d-2k}(A) \leqslant P\{W(I) \cap A \neq \emptyset\} \leqslant C\ \mathrm{Cap}_{d-2k}(A).$$

(see also results of F. Hirsch and S. Song (1991, 1995).

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Brownian sheet: polarity of points

Corollary 3

For the Brownian sheet, points are polar if and only if $d \geqslant 2k$.

Remark

(a) The critical dimension is $Q = 2k$, and points are polar in dimension $d = 2k$.

(b) The result of Khoshnevisan and Shi gives lots of additional information, concerning, for instance, Hausdorff dimension of the range, of level sets, bounds on probabilities that the level sets intersect a deterministic set, ...

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Measuring the size of sets: capacity

Capacity. $\text{Cap}_\beta(A)$ denotes the Bessel-Riesz capacity of A:

$$\text{Cap}_\beta(A) = \frac{1}{\inf_{\mu \in \mathcal{P}(A)} \mathcal{E}_\beta(\mu)},$$

$$\mathcal{E}_\beta(\mu) = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} k_\beta(x-y)\mu(dx)\mu(dy)$$

and

$$k_\beta(x) = \begin{cases} \|x\|^{-\beta} & \text{if } 0 < \beta < d, \\ \ln(\frac{1}{\|x\|}) & \text{if } \beta = 0, \\ 1 & \text{if } \beta < 0. \end{cases}$$

Examples. If $A = \{z\}$, then:

$$\text{Cap}_\beta(\{z\}) = \begin{cases} 1 & \text{if } \beta < 0, \\ 0 & \text{if } \beta \geqslant 0. \end{cases}$$

If A is a subspace of $\mathbb{R}^d$ with dimension $\ell \in \{1, \ldots, d-1\}$, then:

$$\text{Cap}_\beta(A) \begin{cases} > 0 & \text{if } \beta < \ell, \\ = 0 & \text{if } \beta \geqslant \ell. \end{cases}$$

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Back to the Brownian sheet

Theorem. [Khoshnevisan and Shi, 1999, upper bound] Fix $M > 0$. Let I be a box. There exists $0 < C < \infty$ such that for all compact sets $A \subset B(0, M)$ $(\subset \mathbb{R}^d)$,

$$P\{W(I) \cap A \neq \emptyset\} \leqslant C \ \mathrm{Cap}_{d-2k}(A).$$

Remark

When $d = 2k$ and $A = \{z\} \subset \mathbb{R}^d$, then the r.h.s. is zero, which implies that points are polar in the critical dimension $d = 2k$.

Another measure of the size of sets: Hausdorff measure

For $\beta \geqslant 0$, the β-dimensional Hausdorff measure of A is defined by

$$\mathcal{H}_\beta(A) = \lim_{\epsilon \to 0^+} \inf \left\{ \sum_{i=1}^{\infty} (2r_i)^\beta : A \subseteq \bigcup_{i=1}^{\infty} B(x_i, r_i),\ \sup_{i \geq 1} r_i \leqslant \epsilon \right\}.$$

When $\beta < 0$, we define $\mathcal{H}_\beta(A)$ to be ∞.

Note. For $\beta_1 > \beta_2 > 0$,

$$\text{Cap}_{\beta_1}(A) > 0 \Rightarrow \mathcal{H}_{\beta_1}(A) > 0 \Rightarrow \text{Cap}_{\beta_2}(A) > 0.$$

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Anisotropic Gaussian fields (Xiao, 2008)

Let $(V(x),\ x \in \mathbb{R}^k)$ be a centered continuous Gaussian random field with values in $\mathbb{R}^d$ with i.i.d. components: $V(x) = (V_1(x), \dots, V_d(x))$. Set

$$\sigma^2(x, y) = E[(V_1(x) - V_1(y))^2].$$

Let I be a "rectangle". Assume the two conditions:

(C1) There exists $0 < c < \infty$ and $H_1, \dots, H_k \in]0, 1[$ such that for all $x \in I$,

$$c^{-1} \leqslant \sigma^2(0, x) \leqslant c,$$

and for all $x, y \in I$,

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(H_j is the Hölder exponent for coordinate j).

(C2) There is $c > 0$ such that for all $x, y \in I$,

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Anisotropic Gaussian fields

Theorem 4 (Biermé, Lacaux & Xiao, 2007)

Fix $M > 0$. Set

$$Q = \sum_{j=1}^{k} \frac{1}{H_j}.$$

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$$C^{-1}\ Cap_{d-Q}(A) \leqslant P\{V(I) \cap A \neq \emptyset\} \leqslant C\mathcal{H}_{d-Q}(A).$$

Special case obtained by D., Khoshnevisan and E. Nualart (2007)

This results tells us what sort of inequality to aim for when we have a non-Gaussian random field and information about its Hölder exponents.

Notice the Hausdorff measure appearing on the right-hand side.

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Hitting points in the critical dimension

If the results of Biermé, Lacaux & Xiao had "capacity" on the r.h.s. instead of Hausdorff measure:

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then when $d = Q$ and $A = \{z\}$ $(z \in \mathbb{R}^d)$, since $\mathrm{Cap}_0(A) = 0$, we would have

$$P\{V(I) \cap A \neq \emptyset\} = 0$$

and this would show that points are polar.

For an anisotropic Gaussian random field V, we only have:

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If $d = Q$ and $A = \{z\}$ $(z \in \mathbb{R}^d)$, then $\mathcal{H}_0(A) = 1$, so polarity of points remains unclear.

(This issue can be decided on a case-by-case basis for many Gaussian processes.)

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Funaki's random string

Let $(u(t,x),\ (t,x) \in \mathbb{R}_+ \times \mathbb{R})$ be an $\mathbb{R}^d$-valued random field such that

$$\frac{\partial}{\partial t} u(t,x) = \frac{\partial^2}{\partial x^2} u(t,x) + \dot{W}(t,x), \qquad x \in \mathbb{R},\ t > 0,$$

$u(0,\cdot) : \mathbb{R} \to \mathbb{R}^d$ given, $\dot{W}(t,x)$ is space-time white noise.

Theorem 5 (Mueller & Tribe, 2002)

The critical dimension for hitting points is $d = 6$ and points are polar in this dimension.

Their proof uses the "stationary pinned string," then scaling and time reversal (method of Paul Lévy).

It does not apply to the wave equation, nor to heat equation with deterministic non-constant coefficients, such as

$$\frac{\partial}{\partial t} u(t,x) = \frac{\partial^2}{\partial x^2} u(t,x) + \sigma(t,x)\dot{W}(t,x),$$

where $(t,x) \mapsto \sigma(t,x)$ is deterministic but not constant.

(They also treat the issue of double points for this random field)

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Polarity of points in dimensions > the critical dimension

Case $k = 1$, $d \geqslant 3$: let $(B(t), t \in \mathbb{R}_+)$ be a standard Brownian motion with values in $\mathbb{R}^3$. Want to explain why it does not hit points.

Explanation. Let $t_k = 1 + k2^{-2n}$. Fix $x \in \mathbb{R}^d$. Then

$$
\begin{aligned}
P\{\exists t \in [1,2] : B(t) = x\} &= P\left(\bigcup_{k=1}^{2^{2n}} \{\exists t \in [t_{k-1}, t_k] : B(t) = x\}\right) \\
&\leqslant \sum_{k=1}^{2^{2n}} P\{\exists t \in [t_{k-1}, t_k] : B(t) = x\} \\
&= \sum_{k=1}^{2^{2n}} \left[P\{\|B_{t_k} - x\| \leqslant n2^{-n}\} + e(n)\right] \\
&\leqslant \sum_{k=1}^{2^{2n}} \left[c(n2^{-n})^d + e(n)\right] \\
&= 2^{2n}\left[c(n2^{-n})^d + e(n)\right] \\
&= cn^d 2^{(2-d)n} + 2^{2n} e(n) \\
&\to 0 \qquad \text{as } n \to +\infty \text{ (because } d \geqslant 3).
\end{aligned}
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Polarity of points in dimensions > the critical dimension

Case $k = 1$, $d \geqslant 3$: let $(B(t), t \in \mathbb{R}_+)$ be a standard Brownian motion with values in $\mathbb{R}^3$. Want to explain why it does not hit points.

Explanation. Let $t_k = 1 + k2^{-2n}$. Fix $x \in \mathbb{R}^d$. Then

$$
\begin{aligned}
P\{\exists t \in [1,2] : B(t) = x\} &= P\left(\bigcup_{k=1}^{2^{2n}} \{\exists t \in [t_{k-1}, t_k] : B(t) = x\}\right) \\
&\leqslant \sum_{k=1}^{2^{2n}} P\{\exists t \in [t_{k-1}, t_k] : B(t) = x\} \\
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Hitting probabilities for non-Gaussian processes: upper bounds

Let $U = \{U(t,x),\ (t,x) \in \mathbb{R}_+ \times \mathbb{R}^k\}$ be an $\mathbb{R}^d$-valued continuous process.

Theorem 6 (D. & Sanz-Solé, 2015: upper bound)

Let $D \subset \mathbb{R}^d$. Assume that:
(1) For any $x \in \mathbb{R}^k$, $U(t,x)$ has a density $p_{(t,x)}$, and

$$\sup_{z \in D^{(2)}} \sup_{(t,x) \in (I \times J)^{(1)}} p_{(t,x)}(z) \leq C$$

($D^{(2)}$ is the 2-enlargement of D.)

(2) There exist $H_1, H_2 \in \,]0,1]$ and a constant C such that, for any $q \in [1, \infty[$, $(t,x), (s,y) \in (I \times J)^{(1)}$,

$$E\left(\|U(t,x) - U(s,y))\|^q\right) \leq C(|t-s|^{H_1} + \|x-y\|^{H_2})^q.$$

Then for any $\eta > 0$, for every Borel set $A \subset D$,

$$P\{v(I) \cap A \neq \emptyset\} \leq C\mathcal{H}_{d-\eta-\frac{1}{H_1}-\frac{k}{H_2}}(A).$$

Remarks. (a) Condition (2) is essentially a condition on Hölder continuity.
(b) Condition (1) can often be obtained by using Malliavin calculus.
(c) Note that $d - \eta$ appears in the upper bound, but this is otherwise similar to the result of Biermé, Lacaux and Xiao (2007).

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Non-polarity of points in dimensions < the critical dimension

Let $(W(x),\ x \in \mathbb{R}_+^k)$ be a k-parameter $\mathbb{R}^d$-valued Brownian sheet.
Want to show that for $d < 2k$, a point $z \in \mathbb{R}^d$ is not polar. Set $I = [1,2]^k$.

$$P\{\exists x \in I : W(x) = z\} = \lim_{\varepsilon \downarrow 0} P\{W(I) \cap B(z,\varepsilon) \neq \emptyset\}.$$

Define $J_\varepsilon = \int_I dx\, 1_{B(z,\varepsilon)}(W(x))$. Then

$$P\{W(I) \cap B(z,\varepsilon) \neq \emptyset\} \geqslant P\{J_\varepsilon > 0\} \geqslant \frac{(E(J_\varepsilon))^2}{E(J_\varepsilon^2)} \geqslant \frac{c^2}{C} > 0.$$

Lower bound on $E(J_\varepsilon)$:

$$E(J_\varepsilon) = \int_I dx\ P\{W(x) \in B(z,\varepsilon)\} \geqslant \int_I dx\ \varepsilon^d \inf_{w \in B(z,\varepsilon)} p_x(w) = c\,\varepsilon^d.$$

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Need: (1) a lower bound on the probability density function of $W(x)$;
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Hitting probabilities for non-Gaussian processes: lower bounds

Let $U = \{U(x),\ x \in \mathbb{R}^k\}$, $k \in \mathbb{N}^*$, be an $\mathbb{R}^d$-valued continuous process.

Theorem 7 (D. & Sanz-Solé, 2015: lower bound)

Fix $N > 0$, $I \subset \mathbb{R}^k$ compact with positive Lebesgue measure, and assume:
(1) The density $p_x(\cdot)$ of $U(x)$ is continuous, bounded, and positive.
(2) For any $x, y \in I$ with $x \neq y$, $(U(x), U(y))$ has a density $p_{x,y}(\cdot, \cdot)$ w.r.t. Lebesgue measure in $\mathbb{R}^{2d}$, and there exist $\gamma, \alpha \in \,]0, \infty[$ such that for any $z_1, z_2 \in [-N, N]^d$

$$p_{x,y}(z_1, z_2) \leqslant \frac{C}{\|x - y\|^{\gamma}} \left[\frac{\|x - y\|^{\alpha}}{\|z_1 - z_2\|} \wedge 1 \right]^p,$$

where $p > (\gamma - k)\frac{2d}{\alpha} \vee 2$. Then there exists $c > 0$ such that for all Borel sets $A \subset [-N, N]^d$,

$$P\{U(I) \cap A \neq \emptyset\} \geq c \ \mathrm{Cap}_{\frac{1}{\alpha}(\gamma - k)}(A).$$

Remark. The r.h.s. in (2) is not of Gaussian type. It is a weaker condition.

Non-linear systems of stochastic p.d.e.'s

Let L be a partial differential operator (e.g. $L = \frac{\partial}{\partial t} - \Delta$ or $L = \frac{\partial^2}{\partial t^2} - \Delta$).

Let $u(t,x) = (u^1(t,x), \ldots, u^d(t,x)) \in \mathbb{R}^d$ be the solution of

$$\begin{cases} Lu^1(t,x) &= b^1(u(t,x)) + \sum_{j=1}^d \sigma_{1,j}(u(t,x)) \dot{W}_j(t,x), \\ &\vdots \\ Lu^d(t,x) &= b^d(u(t,x)) + \sum_{j=1}^d \sigma_{d,j}(u(t,x)) \dot{W}_j(t,x), \end{cases}$$

$$t \in \,]0,T], \quad x \in \mathbb{R}^k.$$

smooth (Lipschitz) non-linearities: $b^i, \; \sigma_{i,j} : \mathbb{R}^d \to \mathbb{R}, \qquad i = 1, \ldots, d$

Initial conditions: e.g. $u(0,x) = u_0(x)$ given.

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Nonlinear systems of spde's

Suppose that we have optimal Hölder exponents for the solution:

$$c(p)\,\Delta(t,x;s,y) \leqslant \|u(t,x) - u(s,y)\|_{L^p} \leqslant C(p)\,\Delta(t,x;s,y),$$

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$$\Delta(t,x;s,y) = |t-s|^{H_1} + \|x-y\|^{H_2}.$$

Define

$$Q = \frac{1}{H_1} + \frac{k}{H_2}.$$

Typical result 8

Fix $\eta > 0$. Then

$$c_\eta\, Cap_{d-Q+\eta}(A) \leqslant P\{u(I \times J) \cap A \neq \emptyset\} \leqslant C_\eta \mathcal{H}_{d-Q-\eta}(A)$$

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Cases considered

Wave equation, $k = 1$ (D. & E. Nualart, 2004):

$$Lu^i(t,x) = \frac{\partial^2 u^i}{\partial t^2}(t,x) - \frac{\partial^2 u^i}{\partial x^2}(t,x)$$

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Results for systems of non-linear equations

Heat equation, $k = 1$, space-time white noise

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$H_1 = \frac{1}{4}$, $H_2 = \frac{1}{2}$, $\gamma = d + \eta$ $(\eta > 0)$, $Q = \left(\frac{1}{4}\right)^{-1} + \left(\frac{1}{2}\right)^{-1} = 6$,

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Results for systems of non-linear equations

Heat equation, $k = 1$, space-time white noise

[D., Khoshnevisan & E. Nualart, 2007, 2009]

$H_1 = \frac{1}{4}$, $H_2 = \frac{1}{2}$, $\gamma = d + \eta$ $(\eta > 0)$, $Q = \left(\frac{1}{4}\right)^{-1} + \left(\frac{1}{2}\right)^{-1} = 6$,

$$c_\eta \operatorname{Cap}_{d+\eta-6}(A) \leqslant P\{u(I \times J) \cap A \neq \emptyset\} \leqslant C_\eta \mathcal{H}_{d-\eta-6}(A)$$

Heat equation, $k \geqslant 1$, spatially homogeneous noise:

$$E[\dot{W}_i(t,x)\dot{W}_j(s,y)] = \delta(t-s) \|x-y\|^{-\beta} \delta_{ij},$$

[D., Khoshnevisan & E. Nualart, 2013]

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Wave equation, $k \in \{1, 2, 3\}$, spatially homogeneous noise

[D., Sanz-Solé, MAMS, 2015)]

$H_1 = H_2 = \frac{2-\beta}{2}$, $\gamma = d + \eta + \frac{4d^2}{2-\beta}$ $(\eta > 0)$, $Q = \frac{1}{H_1} + \frac{k}{H_2}$,

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Polarity in non-critical dimensions

Let $Q = \frac{1}{H_1} + \frac{k}{H_2}$, where H_1 and H_2 are the optimal Hölder exponents.

Corollary 9 (Polarity of points)

(a) For the systems of stochastic heat equations, points are polar if $d > Q$, and are not polar if $d < Q$.

(b) For systems of stochastic wave equations,

- *if $k \in \{1, 2\}$, then points are polar if $d > Q$, and are not polar if $d < Q$.*
- *if $k = 3$, the points are polar if $d > Q$, and are not polar if $d < Q - \frac{4d^2}{2-\beta}$.*

Towards polarity in critical dimensions

Builds on works of Talagrand (1995, 1998) for fBm.
Let $v = (v(t,x),\ t \geqslant 0,\ x \in \mathbb{R}^k)$ be $\mathbb{R}^d$-valued Gaussian random field.

Assumption 10

There is a random field $(V(A,t,x),\ A \in \mathcal{B}(\mathbb{R}_+),\ t \geqslant 0,\ x \in \mathbb{R}^k)$ *such that:*

(a) for (t,x) *fixed,* $A \mapsto V(A,t,x)$ *is an* $\mathbb{R}^d$*-valued Gaussian white noise;*
(b) when $A \cap B = \emptyset$, $V(A,\cdot,\cdot)$ *and* $V(B,\cdot,\cdot)$ *are independent;*
(c) non-degeneracy assumptions;

(d) $\exists c > 0,\ \gamma_1 > 0,\ \gamma_2 > 0$ *such that:*

$$\|v(t,x) - v(s,y) - (V([a,b[,t,x) - V([a,b[,s,y))\|_{L^2} \leqslant c\left[a^{\gamma_1}|t-s| + a^{\gamma_2}\|x-y\| + b^{-1}\right]$$

Remarks. (a) The γ_j are related to the Hölder exponents: $\gamma_j = \frac{1}{H_j} - 1$.
(b) Condition (d) states that if $|t-s| \sim 2^{-n/H_1}$ and $\|x-y\| \sim 2^{-n/H_2}$, then

$$\|v(t,x) - v(s,y)\| \sim \|V([2^n, 2^{n+1}[,t,x) - V([2^n, 2^{n+1}[,s,y)\|$$

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Polarity in critical dimensions

Theorem 11 (D., Mueller & Xiao, AoP, 2017?)

Under Assumption 10, if $d = Q := \frac{1}{H_1} + \frac{k}{H_2}$, then for all $z \in \mathbb{R}^d$,

$$P\{\exists(t,x) : v(t,x) = z\} = 0,$$

that is, points are polar (and $d = Q$ is the critical dimension).

Question. For which linear systems of s.p.d.e.'s is Assumption 10 satisfied ?

Proposition 12

Assumption 10 is satisfied in the following cases:

Heat equation, $k = 1$, space-time white noise

Wave equation, $k = 1$, space-time white noise

Heat equation, $k \geqslant 1$, spatially homogeneous noise

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Example

Heat equations, spatial dimension 1, space-time white noise

Let $v = (v(t,x),\ t \in \mathbb{R}_+,\ x \in \mathbb{R})$ solve

$$\begin{cases} \frac{\partial}{\partial t} v_j(t,x) = \frac{\partial^2}{\partial x^2} v_j(t,x) + \dot{W}_j(t,x), & j = 1,\dots,d, \\ v(0,x) = 0, & x \in \mathbb{R}. \end{cases}$$

Corollary 13

Suppose $d = 6$ (critical dimension). Then points are polar for v.

Remark. We also get (essentially for free) that there are no double points in the critical dimension $d = 12$, simply by considering the process

$$\tilde{v}(t,x,s,y) := v(t,x) - v(s,y),$$

which also satisfies Assumption 10, but with twice as many parameters.

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Expressions for the solution of the stochastic heat equation

Standard expression for $v(t,x)$. Let

$$G(s,y) = (4\pi t)^{-1/2} \exp[-y^2/(4t)].$$

Then

$$v(t,x) = \int_{[0,t]\times\mathbb{R}} G(t-s, x-y)\, \hat{W}(ds, dy).$$

Another expression for $v(t,x)$. Use space-time Fourier transform and Plancherel's theorem:

$$v(t,x) = \langle \dot{\hat{W}},\ G(t-\cdot, x-\cdot)\rangle = \langle \mathcal{F}\dot{\hat{W}},\ \mathcal{F}G^{\vee}(t-\cdot, x-\cdot)\rangle = \langle \text{white noise},\ F_{(t,x)}\rangle$$

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$$F_{(t,x)}(\tau,\xi) = \mathcal{F}_{s,y} G^{\vee}(t-\cdot, x-\cdot)(\tau,\xi) = e^{-i\xi x} \frac{e^{-i\tau t} - e^{-t|\xi|^2}}{|\xi|^2 - i\tau}$$

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Harmomizable representation of the solution

The solution can be written:

$$v(t,x) = \int_{\mathbb{R}} \int_{\mathbb{R}} e^{-i\xi\cdot x} \frac{e^{-i\tau t} - e^{-t|\xi|^2}}{|\xi|^2 - i\tau} W(d\tau, d\xi).$$

(also appears in Balan, 2012).

Assumption 10 is satisfied by setting

$$V(A,t,x) := \iint_{\max(|\tau|^{\frac{1}{4}},\, |\xi|^{\frac{1}{2}}) \in A} e^{-i\xi\cdot x} \frac{e^{-i\tau t} - e^{-t|\xi|^2}}{|\xi|^2 - i\tau} W(d\tau, d\xi).$$

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Proving the inequality requires estimating some double integrals.

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Conclusions

- There are now good sufficient conditions for upper bounds on hitting probabilities.
- Some Malliavin calculus is needed if the process is not Gaussian.
- These conditions imply polarity of points in non-critical dimensions.
- There are now good sufficient conditions for lower bounds on hitting probabilities.
- These also require Malliavin calculus in the non-Gaussian case, and checking the conditions can be difficult (e.g. wave equation).
- For a wide class of Gaussian random fields, there are good sufficient conditions for polarity of points in critical dimensions.
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