Volatility and Arbitrage

Johannes Ruf

London School of Economics

Opening Conference of the Thematic Semester
on "Stochastic Modeling"
Verona
December 2017

Joint work with Bob Fernholz and Ioannis Karatzas

Outline

Stochastic Portfolio Theory (SPT)

A rich and flexible framework introduced by Bob Fernholz for analyzing portfolio behavior and equity market structure.

Research in SPT focuses on two main areas.

1. Abstract markets: building models that reflect properties of real equity markets.
2. Arithmetics of returns: relevance of logarithmic returns, the role of diversification, constructing relative arbitrages.

General setup

- A probability space $(\Omega, \mathscr{F}, \mathrm{P})$ equipped with a right-continuous filtration $\mathfrak{F}$.
- $d \in \mathbb{N}$: number of assets at time zero. E.g., $d = 505$ (S&P 500) or $d = 4152$.

General setup

- A probability space $(\Omega, \mathscr{F}, \mathrm{P})$ equipped with a right-continuous filtration $\mathfrak{F}$.
- $d \in \mathbb{N}$: number of assets at time zero. E.g., $d = 505$ (S&P 500) or $d = 4152$.
- Nonnegative continuous semimartingales $S_1(\cdot), \cdots, S_d(\cdot)$, representing the capitalization (stock-price, multiplied by the number of shares outstanding) of each company.
- For example, $S_i(\cdot)$ might be an Itô process of the form

$$\mathrm{d}S_i(t) = S_i(t)\left[b_i(t)\mathrm{d}t + \sum_{\nu=1}^{N} \sigma_{i,\nu}(t)\mathrm{d}W_\nu(t)\right],$$

  where $W(\cdot)$ denotes an N–dimensional vector of independent Brownian motions.
- For simplicity, there is no traded bond.

Market weights

- Let $\Sigma(t)$ denote the total market capitalization at time t; i.e.:

$$\Sigma(t) = S_1(t) + \cdots + S_d(t).$$

- We shall assume, throughout, that $S_1(t) + \cdots + S_d(t) > 0$.

Market weights

- Let $\Sigma(t)$ denote the total market capitalization at time t; i.e.:

$$\Sigma(t) = S_1(t) + \cdots + S_d(t).$$

- We shall assume, throughout, that $S_1(t) + \cdots + S_d(t) > 0$.
- Then the relative market weights $\mu_1(\cdot), \cdots, \mu_d(\cdot)$ of each asset are given by

$$\mu_i(t) = \frac{S_i(t)}{\Sigma(t)}$$

and take values in

$$\boldsymbol{\Delta}^d = \left\{ \left(x_1, \cdots, x_d\right)' \in [0,1]^d \; : \; \sum_{i=1}^{d} x_i = 1 \right\}.$$

An important empirical property of equity markets

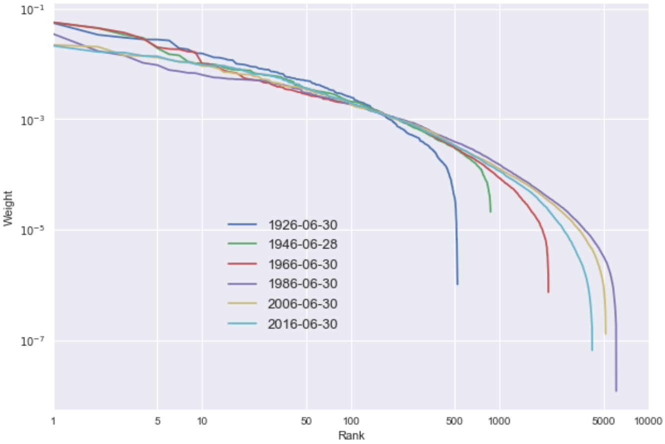


Figure: The capital distribution curve — Market weights $\mu_i(\cdot)$ against ranks on logarithmic scale, 1926–2016.

Abstract market models

- Not easy to write down tractable mathematical models for $S_1(\cdot), \cdots, S_d(\cdot)$ whose capital market curves (and especially their dynamics) resemble the empirical ones.
- For one-dimensional stock price dynamics, very helpful models have been developed with realistic dynamics. (Samuelson-Black-Scholes-Merton, stochastic volatility, rough volatility, ...).
- Unfortunately, just combining such models does not yield realistic market models.

Outline

Returns: An MBA overview

- The classical definition of return on an investment is

$$\text{return} = \frac{\text{final value} \; - \; \text{initial value}}{\text{initial value}}.$$

Returns: An MBA overview

- The classical definition of return on an investment is

$$\text{return} = \frac{\text{final value} - \text{initial value}}{\text{initial value}}.$$

- Suppose we wish to calculate the average annual return of an investment over several years, where the annual returns are given by $r_1, r_2, \ldots, r_n$.

Returns: An MBA overview

- The classical definition of return on an investment is

$$\text{return} = \frac{\text{final value} - \text{initial value}}{\text{initial value}}.$$

- Suppose we wish to calculate the average annual return of an investment over several years, where the annual returns are given by $r_1, r_2, \ldots, r_n$.
- Several common methods are available.
  1. *Arithmetic return:* $\frac{1}{n}\Big((1 + r_1) + \cdots + (1 + r_n)\Big) - 1.$
  2. *Geometric return:* $\sqrt[n]{(1 + r_1) \times \cdots \times (1 + r_n)} - 1.$
  3. *Logarithmic return:* $\frac{1}{n}\Big(\log(1 + r_1) + \cdots + \log(1 + r_n)\Big).$

Some remarks on computing the average returns

1. *Arithmetic return:* $\frac{1}{n}\Big((1+r_1)+\cdots+(1+r_n)\Big)-1.$
   Used in Modern Portfolio Theory. Compatible with the linear models used to calculate the Sharpe ratio and beta. But leads to absurd estimates in some cases.

Some remarks on computing the average returns

1. *Arithmetic return:* $\frac{1}{n}\Big((1+r_1)+\cdots+(1+r_n)\Big)-1.$
   Used in Modern Portfolio Theory. Compatible with the linear models used to calculate the Sharpe ratio and beta. But leads to absurd estimates in some cases.
2. *Geometric return:* $\sqrt[n]{(1+r_1)\times\cdots\times(1+r_n)}-1.$
   Sometimes very difficult to compute.

Some remarks on computing the average returns

1. *Arithmetic return:* $\frac{1}{n}\Big((1+r_1)+\cdots+(1+r_n)\Big)-1$.
   Used in Modern Portfolio Theory. Compatible with the linear models used to calculate the Sharpe ratio and beta. But leads to absurd estimates in some cases.
2. *Geometric return:* $\sqrt[n]{(1+r_1)\times\cdots\times(1+r_n)}-1$.
   Sometimes very difficult to compute.
3. *Logarithmic return:* $\frac{1}{n}\Big(\log(1+r_1)+\cdots+\log(1+r_n)\Big)$. Used in stochastic portfolio theory.

Some remarks on computing the average returns

1. *Arithmetic return:* $\frac{1}{n}\Big((1+r_1)+\cdots+(1+r_n)\Big)-1$.
   Used in Modern Portfolio Theory. Compatible with the linear models used to calculate the Sharpe ratio and beta. But leads to absurd estimates in some cases.
2. *Geometric return:* $\sqrt[n]{(1+r_1)\times\cdots\times(1+r_n)}-1$.
   Sometimes very difficult to compute.
3. *Logarithmic return:* $\frac{1}{n}\Big(\log(1+r_1)+\cdots+\log(1+r_n)\Big)$. Used in stochastic portfolio theory.

Jensen's inequality yields

$$\text{arithmetic return} \;\geq\; \text{geometric return} \;\geq\; \text{logarithmic return}.$$

The dynamics of return

Let $S(t)$ represent the price of a stock at time t. Assume that

$$\mathrm{d}S(t) = S(t)\Big[b(t)\mathrm{d}t + \sigma(t)\mathrm{d}W(t)\Big].$$

Then b is called the *rate of return* of S.

The dynamics of return

Let $S(t)$ represent the price of a stock at time t. Assume that

$$\mathrm{d}S(t) = S(t)\Big[b(t)\mathrm{d}t + \sigma(t)\mathrm{d}W(t)\Big].$$

Then b is called the *rate of return* of S.

- Itô's formula implies that

$$\mathrm{d}\log S(t) = g(t)\,\mathrm{d}t + \sigma(t)\mathrm{d}W(t),$$

  where

$$g(t) = b(t) - \frac{1}{2}\sigma^2(t).$$

  g is called the *rate of log-return*, or *growth rate*, of S.

The dynamics of return

Let $S(t)$ represent the price of a stock at time t. Assume that

$$\mathrm{d}S(t) = S(t)\Big[b(t)\mathrm{d}t + \sigma(t)\mathrm{d}W(t)\Big].$$

Then b is called the *rate of return* of S.

- Itô's formula implies that

$$\mathrm{d}\log S(t) = g(t)\,\mathrm{d}t + \sigma(t)\mathrm{d}W(t),$$

  where

$$g(t) = b(t) - \frac{1}{2}\sigma^2(t).$$

  g is called the *rate of log-return*, or *growth rate*, of S.

- The process g determines the long-term behavior of S:

$$\lim_{T\uparrow\infty} \frac{1}{T}\Big(\log S(T) - \int_0^T g(t)\mathrm{d}t\Big) = 0$$

  (under appropriate assumptions).

Portfolio return and log-return

Suppose we have assets $S_1, \ldots, S_d$ and a portfolio π with weights $\pi_1(t) + \cdots + \pi_d(t) = 1$ and value $V^\pi(t)$ at time t. Then the portfolio return satisfies

$$\frac{\mathrm{d}V^\pi(t)}{V^\pi(t)} = \sum_i \pi_i(t) \frac{\mathrm{d}S_i(t)}{S_i(t)}$$

(Markowitz (1952)).

Portfolio return and log-return

Suppose we have assets $S_1, \ldots, S_d$ and a portfolio π with weights $\pi_1(t) + \cdots + \pi_d(t) = 1$ and value $V^\pi(t)$ at time t. Then the portfolio return satisfies

$$\frac{\mathrm{d}V^\pi(t)}{V^\pi(t)} = \sum_i \pi_i(t) \frac{\mathrm{d}S_i(t)}{S_i(t)}$$

(Markowitz (1952)).
The analogous equation for log-return is

$$\mathrm{d}\log V^\pi(t) = \sum_i \pi_i(t)\,\mathrm{d}\log S_i(t) + \gamma^*_\pi(t)\,\mathrm{d}t,$$

where $\gamma^*_\pi \geq 0$, provided the portfolio is long-only. (Fernholz and Shay (1982)).

Portfolio return and log-return

Suppose we have assets $S_1, \ldots, S_d$ and a portfolio π with weights $\pi_1(t) + \cdots + \pi_d(t) = 1$ and value $V^\pi(t)$ at time t. Then the portfolio return satisfies

$$\frac{\mathrm{d}V^\pi(t)}{V^\pi(t)} = \sum_i \pi_i(t) \frac{\mathrm{d}S_i(t)}{S_i(t)}$$

(Markowitz (1952)).
The analogous equation for log-return is

$$\mathrm{d}\log V^\pi(t) = \sum_i \pi_i(t)\,\mathrm{d}\log S_i(t) + \gamma^*_\pi(t)\,\mathrm{d}t,$$

where $\gamma^*_\pi \geq 0$, provided the portfolio is long-only. (Fernholz and Shay (1982)).

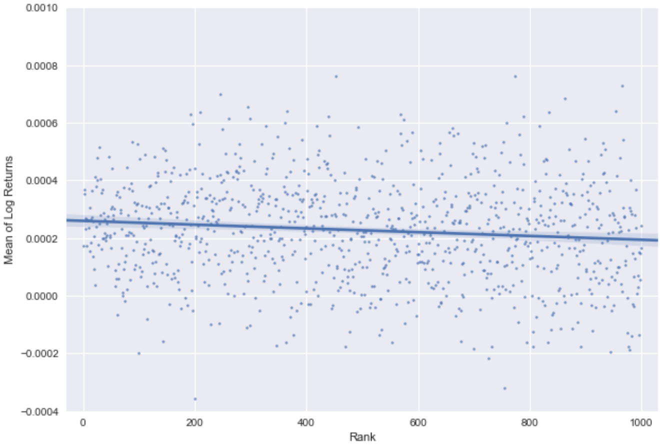
γ^*_π **depends only on the covariance structure of** S**.**

Rank-based analysis of logarithmic returns

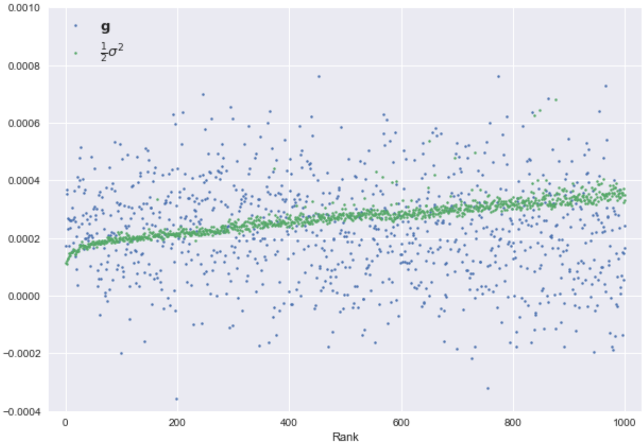
- Let $r_t(i)$ be the rank of $S_i(t)$.
- Define the *average rank-based growth rates* $\boldsymbol{g}_k$ over $[0, T]$ by

$$\boldsymbol{g}_k = \frac{1}{T} \int_0^T \sum \mathbf{1}_{\{r_t(i)=k\}} \mathrm{d} \log S_i(t).$$

Estimated g_k, 1962–2016



Including sample variance



Outline

Recalling the setup

- $d \in \mathbb{N}$: number of assets at time zero.
- Nonnegative continuous semimartingales $S_1(\cdot), \cdots, S_d(\cdot)$, the capitalizations, such that $\sum_i S_i(\cdot) > 0$.
- No bond.

Recalling the setup

- $d \in \mathbb{N}$: number of assets at time zero.
- Nonnegative continuous semimartingales $S_1(\cdot), \cdots, S_d(\cdot)$, the capitalizations, such that $\sum_i S_i(\cdot) > 0$.
- No bond.
- Relative market weights $\mu_1(\cdot), \cdots, \mu_d(\cdot)$, given by

$$\mu_i(t) = \frac{S_i(t)}{\Sigma(t)}, \qquad \text{where} \qquad \Sigma(t) = S_1(t) + \cdots + S_d(t),$$

and taking values in

$$\mathbf{\Delta}^d = \left\{ \left(x_1, \cdots, x_d\right)' \in [0,1]^d \; : \; \sum_{i=1}^{d} x_i = 1 \right\}.$$

Recalling the setup

- $d \in \mathbb{N}$: number of assets at time zero.
- Nonnegative continuous semimartingales $S_1(\cdot), \cdots, S_d(\cdot)$, the capitalizations, such that $\sum_i S_i(\cdot) > 0$.
- No bond.
- Relative market weights $\mu_1(\cdot), \cdots, \mu_d(\cdot)$, given by

$$\mu_i(t) = \frac{S_i(t)}{\Sigma(t)}, \qquad \text{where} \qquad \Sigma(t) = S_1(t) + \cdots + S_d(t),$$

  and taking values in

$$\boldsymbol{\Delta}^d = \left\{ \big(x_1, \cdots, x_d\big)' \in [0,1]^d \ : \ \sum_{i=1}^{d} x_i = 1 \right\}.$$

- We can think of $\mu_i(t)$ as the "price" of the i-th company, expressed in terms of the total market capitalization.

Recalling the setup

- $d \in \mathbb{N}$: number of assets at time zero.
- Nonnegative continuous semimartingales $S_1(\cdot), \cdots, S_d(\cdot)$, the capitalizations, such that $\sum_i S_i(\cdot) > 0$.
- No bond.
- Relative market weights $\mu_1(\cdot), \cdots, \mu_d(\cdot)$, given by

$$\mu_i(t) = \frac{S_i(t)}{\Sigma(t)}, \qquad \text{where} \qquad \Sigma(t) = S_1(t) + \cdots + S_d(t),$$

and taking values in

$$\boldsymbol{\Delta}^d = \left\{ \big(x_1, \cdots, x_d\big)' \in [0,1]^d \ : \ \sum_{i=1}^{d} x_i = 1 \right\}.$$

- We can think of $\mu_i(t)$ as the "price" of the i-th company, expressed in terms of the total market capitalization.
- No frictions; in particular, "small investor" and no trading costs (!)

The role of volatility

- Volatility in an equity market can generate (relative) arbitrage, at least under idealized conditions on market structure and on trading.

The role of volatility

- Volatility in an equity market can generate (relative) arbitrage, at least under idealized conditions on market structure and on trading.
- What level of volatility is required and how long might it take for this arbitrage to be realized?

The role of volatility

- Volatility in an equity market can generate (relative) arbitrage, at least under idealized conditions on market structure and on trading.
- What level of volatility is required and how long might it take for this arbitrage to be realized?
- A common but restrictive condition regarding market volatility, sometimes known as *strict nondegeneracy*, is the requirement that the eigenvalues of the market covariation matrix be bounded away from zero.

The role of volatility

- Volatility in an equity market can generate (relative) arbitrage, at least under idealized conditions on market structure and on trading.
- What level of volatility is required and how long might it take for this arbitrage to be realized?
- A common but restrictive condition regarding market volatility, sometimes known as *strict nondegeneracy*, is the requirement that the eigenvalues of the market covariation matrix be bounded away from zero.
- This is a quite restrictive assumption regarding the behavior over time of the smallest eigenvalue of a random $d \times d$ matrix, where $d \in \mathbb{N}$ is usually a large integer (the number of stocks in an equity market).

A concave transformation

- Consider a concave C^2–function $G : \mathbf{\Delta}^d \to \mathbb{R}$.
- By Itô's formula:

A concave transformation

- Consider a concave C^2–function $G : \mathbf{\Delta}^d \to \mathbb{R}$.
- By Itô's formula:

$$G(\mu(t)) - G(\mu(0)) + \Gamma^G(t) = \int_0^t \langle \vartheta(s), \mathrm{d}\mu(s) \rangle,$$

where $\vartheta(s) = DG(\mu(s))$ and

$$\Gamma^G(t) = -\frac{1}{2} \sum_{i=1}^d \sum_{j=1}^d \int_0^t D_{ij}^2 G\big(\mu(s)\big) \, \mathrm{d}\langle \mu_i, \mu_j \rangle(s).$$

A concave transformation

- Consider a concave C^2–function $G : \boldsymbol{\Delta}^d \to \mathbb{R}$.
- By Itô's formula:

$$G(\mu(t)) - G(\mu(0)) + \Gamma^G(t) = \int_0^t \langle \vartheta(s), \mathrm{d}\mu(s) \rangle,$$

where $\vartheta(s) = DG(\mu(s))$ and

$$\Gamma^G(t) = -\frac{1}{2} \sum_{i=1}^d \sum_{j=1}^d \int_0^t D^2_{ij} G\big(\mu(s)\big)\, \mathrm{d}\langle \mu_i, \mu_j \rangle(s).$$

- The process $\Gamma^G(\cdot)$ is an aggregated cumulative measure of the market's internal variation.
- We shall formulate conditions on this process $\Gamma^G(\cdot)$ (instead of on the $(d-1)$–smallest eigenvalue of the covariation matrix).

Example: entropy function

- Consider the (nonnegative) Gibbs *entropy function*

$$H(x) = \sum_{j=1}^{d} x_j \log\left(\frac{1}{x_j}\right).$$

Example: entropy function

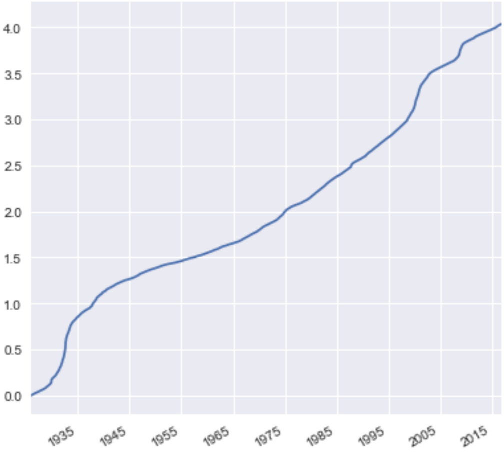
- Consider the (nonnegative) Gibbs *entropy function*

$$H(x) = \sum_{j=1}^{d} x_j \log\left(\frac{1}{x_j}\right).$$

- Assuming $\mu(\cdot) \in \mathbf{\Delta}_+^d$ we have

$$\begin{aligned}\Gamma^H(\cdot) &= \frac{1}{2}\sum_{j=1}^{d}\int_0^{\cdot} \frac{\mathrm{d}\langle \mu_j, \mu_j\rangle(t)}{\mu_j(t)} \\ &= \frac{1}{2}\sum_{j=1}^{d}\int_0^{\cdot} \mu_j(t)\mathrm{d}\langle \log(\mu_j), \log(\mu_j)\rangle(t).\end{aligned}$$

The process $\Gamma^H(\cdot)$



Trading strategies

For an $\mathbb{R}^d$–valued predictable process $\vartheta(\cdot)$ write

$$V^{\vartheta}(t) := \sum_{i=1}^{d} \vartheta_i(t)\mu_i(t).$$

Trading strategies

For an $\mathbb{R}^d$–valued predictable process $\vartheta(\cdot)$ write

$$V^{\vartheta}(t) := \sum_{i=1}^{d} \vartheta_i(t)\mu_i(t).$$

Definition

Suppose that $\vartheta(\cdot)$ is integrable with respect to $\mu(\cdot)$ and that

$$V^{\vartheta}(T) - V^{\vartheta}(0) = \int_0^T \langle \vartheta(t), \mathrm{d}\mu(t) \rangle$$

holds. Then $\vartheta(\cdot)$ is called trading strategy.

Arbitrage relative to the market

Definition

A trading strategy $\vartheta(\cdot)$ is a relative arbitrage with respect to the market portfolio over the time horizon $[0, T]$ if

$$V^{\vartheta}(0) = 1; \qquad V^{\vartheta}(t) \geq 0$$

and

$$\mathrm{P}\Big(V^{\vartheta}(T) \geq 1\Big) = 1; \qquad \mathrm{P}\Big(V^{\vartheta}(T) > 1\Big) > 0.$$

Arbitrage over long term horizons

Let $G : \mathbf{\Delta}^d \to [0, \infty)$ be a concave C^2 function and recall the aggregated cumulative measure of the market's internal variation

$$\Gamma^G(t) = -\frac{1}{2} \sum_{i=1}^{d} \sum_{j=1}^{d} \int_0^t D_{ij}^2 G\left(\mu(s)\right) \mathrm{d}\langle \mu_i, \mu_j \rangle(s).$$

Assume that for some constant $\eta > 0$

$$\mathrm{P}\left(\text{the mapping } [0, \infty) \ni t \mapsto \Gamma^G(t) - \eta t \text{ is nondec.}\right) = 1. \quad (*)$$

Arbitrage over long term horizons

Let $G : \mathbf{\Delta}^d \to [0, \infty)$ be a concave C^2 function and recall the aggregated cumulative measure of the market's internal variation

$$\Gamma^G(t) = -\frac{1}{2} \sum_{i=1}^{d} \sum_{j=1}^{d} \int_0^t D_{ij}^2 G\big(\mu(s)\big) \, \mathrm{d}\langle \mu_i, \mu_j \rangle(s).$$

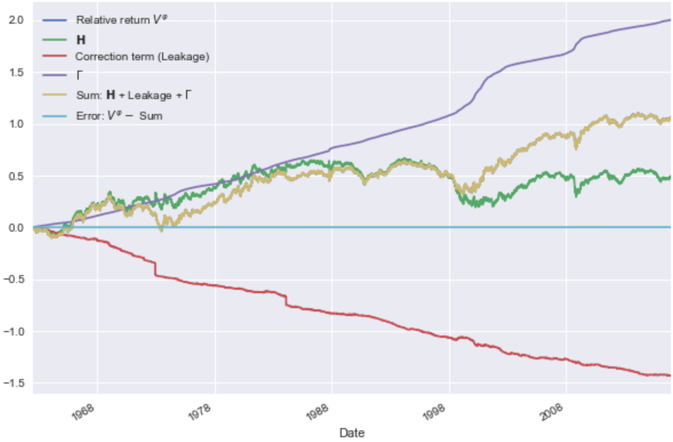
Assume that for some constant $\eta > 0$

$$\mathrm{P}\Big(\text{the mapping } [0, \infty) \ni t \mapsto \Gamma^G(t) - \eta t \text{ is nondec.}\Big) = 1. \quad (*)$$

Theorem

Assume (*). Then there exist $T^* > 0$ and a trading strategy $\varphi^G(\cdot)$ that is relative arbitrage with respect to the market over any time horizon $[0, T]$ with $T \in [T^*, \infty)$.

Cumulative excess growth of the market and relative return



Recall

$$V^{\varphi}(\cdot) = \boldsymbol{H}(\mu(\cdot)) + \Gamma^{\boldsymbol{H}}(\cdot).$$

An important remark

As long as the market model $\mu(\cdot)$ satisfies, for some $\eta > 0$,

$$\mathrm{P}\left(\text{the mapping } [0,\infty) \ni t \mapsto \Gamma^G(t) - \eta t \text{ is nondec.}\right) = 1, \quad (*)$$

the arbitrage strategy

$$\varphi_i^G(\cdot) = D_i G\big(\mu(\cdot)\big) + \Gamma^G(\cdot) + G\big(\mu(\cdot)\big) - \sum_{j=1}^{d} \mu_j(\cdot) D_j G\big(\mu(\cdot)\big).$$

does not depend on the model parameters or the time horizon.

Existence of short-term arbitrage

If it's clear which of the components contributes to the overall variance:

Theorem

Suppose there exists a constant $\eta > 0$ *such that* $\langle \mu_1 \rangle(t) \geq \eta t$ *holds on the stochastic interval* $[\![0, \mathscr{D}^*[\![$ *with*

$$\mathscr{D}^* := \inf \left\{ t \geq 0 : \; \mu_1(t) \leq \frac{\mu_1(0)}{2} \right\}.$$

Then relative arbitrage exists over the time horizon $[0, T]$*, for every* $T > 0$*.*

Existence of short-term arbitrage (cont'd)

If we have full support:

Theorem

Suppose that for a given regular function G and appropriate real constants $\eta > 0$ and $h > 0$,

$$\mathsf{P}\left(\textit{the mapping } [0,\infty) \ni t \mapsto \Gamma^G(t) - \eta t \;\textit{ is nondec.}\right) = 1; \quad (*)$$

$$\mathsf{P}\big(G(\mu(t)) \geq h, \quad t \geq 0\big) = 1$$

and the "time homogeneous support" property

$$\mathsf{P}\Big(G\big(\mu(t)\big) \in \big[h, h{+}\varepsilon\big), \;\textit{ for some } t \in [0,T]\Big) > 0, \quad \textit{for all } T, \varepsilon > 0.$$

Then relative arbitrage exists over the time horizon $[0, T]$, for every $T > 0$.

Existence of short-term arbitrage (cont'd)

Does the condition

$$\mathsf{P}\left(\text{the mapping } [0,\infty) \ni t \mapsto \Gamma^G(t) - \eta t \text{ is nondec.}\right) = 1 \quad (*)$$

always imply the existence of short-term arbitrage?

Answer: No; there exist models that satisfy (*) but do not allow for short-term arbitrage.

A counter-example

- Consider the generating function

$$Q(x) = 1 - \sum_{j=1}^{d} x_j^2.$$

- Then

$$\Gamma^Q(\cdot) = \sum_{j=1}^{d} \langle \mu_j, \mu_j \rangle (\cdot)$$

A counter-example

- Consider the generating function

$$Q(x) = 1 - \sum_{j=1}^{d} x_j^2.$$

- Then

$$\Gamma^Q(\cdot) = \sum_{j=1}^{d} \langle \mu_j, \mu_j \rangle(\cdot)$$

- Goal: Construct process $\mu(\cdot)$ with each component a martingale such that $\Gamma^Q(t) = t$, $t \in [0, T^*]$ for some $T^* > 0$.
- This then yields a counterexample.

An Itô diffusion

- Fix $d = 3$ (three assets).
- Consider SDEs:

$$\mathrm{d}v_1(t) = \frac{1}{\sqrt{3}}(v_2(t) - v_3(t))\mathrm{d}\Theta(t);$$
$$\mathrm{d}v_2(t) = \frac{1}{\sqrt{3}}(v_3(t) - v_1(t))\mathrm{d}\Theta(t);$$
$$\mathrm{d}v_3(t) = \frac{1}{\sqrt{3}}(v_1(t) - v_2(t))\mathrm{d}\Theta(t).$$

An Itô diffusion

- Fix $d = 3$ (three assets).
- Consider SDEs:

$$\mathrm{d}v_1(t) = \frac{1}{\sqrt{3}}(v_2(t) - v_3(t))\mathrm{d}\Theta(t);$$
$$\mathrm{d}v_2(t) = \frac{1}{\sqrt{3}}(v_3(t) - v_1(t))\mathrm{d}\Theta(t);$$
$$\mathrm{d}v_3(t) = \frac{1}{\sqrt{3}}(v_1(t) - v_2(t))\mathrm{d}\Theta(t).$$

- Define $r(v) = \sqrt{\sum_{i=1}^{3} \left(v_i - \frac{1}{3}\right)^2}$.

An Itô diffusion

- Fix $d = 3$ (three assets).
- Consider SDEs:

$$\mathrm{d}v_1(t) = \frac{1}{\sqrt{3}}(v_2(t) - v_3(t))\mathrm{d}\Theta(t);$$
$$\mathrm{d}v_2(t) = \frac{1}{\sqrt{3}}(v_3(t) - v_1(t))\mathrm{d}\Theta(t);$$
$$\mathrm{d}v_3(t) = \frac{1}{\sqrt{3}}(v_1(t) - v_2(t))\mathrm{d}\Theta(t).$$

- Define $r(v) = \sqrt{\sum_{i=1}^{3} \left(v_i - \frac{1}{3}\right)^2}$.
- Itô's formula yields

$$\langle v_1\rangle(t) + \langle v_2\rangle(t) + \langle v_3\rangle(t) = r^2(v(t)) = r^2(v(0))\mathrm{e}^t.$$

An Itô diffusion

- Fix $d = 3$ (three assets).
- Consider SDEs:

$$\mathrm{d}v_1(t) = \frac{1}{\sqrt{3}}(v_2(t) - v_3(t))\mathrm{d}\Theta(t);$$
$$\mathrm{d}v_2(t) = \frac{1}{\sqrt{3}}(v_3(t) - v_1(t))\mathrm{d}\Theta(t);$$
$$\mathrm{d}v_3(t) = \frac{1}{\sqrt{3}}(v_1(t) - v_2(t))\mathrm{d}\Theta(t).$$

- Define $r(v) = \sqrt{\sum_{i=1}^{3}\left(v_i - \frac{1}{3}\right)^2}$.
- Itô's formula yields

$$\langle v_1\rangle(t) + \langle v_2\rangle(t) + \langle v_3\rangle(t) = r^2(v(t)) = r^2(v(0))\mathrm{e}^t.$$

- A solution:

$$v_i(t) = \frac{1}{3} + \delta \mathrm{e}^{t/2}\cos\left(\Theta(t) + 2\pi\left(u + \frac{i-1}{3}\right)\right).$$

An Itô diffusion (cont'd)

- A slight modification:

$$\mathrm{d}v_1(t) = \frac{1}{\sqrt{3}r(t)}(v_2(t) - v_3(t))\mathrm{d}\Theta(t);$$
$$\mathrm{d}v_2(t) = \frac{1}{\sqrt{3}r(t)}(v_3(t) - v_1(t))\mathrm{d}\Theta(t);$$
$$\mathrm{d}v_3(t) = \frac{1}{\sqrt{3}r(t)}(v_1(t) - v_2(t))\mathrm{d}\Theta(t).$$

- This can be made sense of also if

$$v_1(0) = v_2(0) = v_3(0) = 1/3.$$

- Now,

$$\langle v_1\rangle(t) + \langle v_2\rangle(t) + \langle v_3\rangle(t) = r^2(v(t)) = t.$$

- Market model $\mu(\cdot)$: stopped version of $v(\cdot)$.

An Itô diffusion (cont'd)

- A slight modification:

$$\mathrm{d}v_1(t) = \frac{1}{\sqrt{3}r(t)}(v_2(t) - v_3(t))\mathrm{d}\Theta(t);$$
$$\mathrm{d}v_2(t) = \frac{1}{\sqrt{3}r(t)}(v_3(t) - v_1(t))\mathrm{d}\Theta(t);$$
$$\mathrm{d}v_3(t) = \frac{1}{\sqrt{3}r(t)}(v_1(t) - v_2(t))\mathrm{d}\Theta(t).$$

- This can be made sense of also if

$$v_1(0) = v_2(0) = v_3(0) = 1/3.$$

- Now,

$$\langle v_1\rangle(t) + \langle v_2\rangle(t) + \langle v_3\rangle(t) = r^2(v(t)) = t.$$

- Market model $\mu(\cdot)$: stopped version of $v(\cdot)$.

An Itô diffusion (cont'd)

- A slight modification:

$$\mathrm{d}v_1(t) = \frac{1}{\sqrt{3}r(t)}(v_2(t) - v_3(t))\mathrm{d}\Theta(t);$$
$$\mathrm{d}v_2(t) = \frac{1}{\sqrt{3}r(t)}(v_3(t) - v_1(t))\mathrm{d}\Theta(t);$$
$$\mathrm{d}v_3(t) = \frac{1}{\sqrt{3}r(t)}(v_1(t) - v_2(t))\mathrm{d}\Theta(t).$$

- This can be made sense of also if

$$v_1(0) = v_2(0) = v_3(0) = 1/3.$$

- Now,

$$\langle v_1\rangle(t) + \langle v_2\rangle(t) + \langle v_3\rangle(t) = r^2(v(t)) = t.$$

- Market model $\mu(\cdot)$: stopped version of $v(\cdot)$.

Conclusion

For a concave C^2 function G consider the condition

$$\mathsf{P}\left(\text{the mapping } [0,\infty) \ni t \mapsto \Gamma^G(t) - \eta t \text{ is nondec.}\right) = 1. \quad (*)$$

- Under (*), there exists T^* such that relative arbitrage over all $[0, T]$ with $T > T^*$ can be explicitly constructed.
- Under (*) and additional conditions (e.g., time-homogeneous support), for each T there exists relative arbitrage over $[0, T]$.
- Without additional conditions, (*) is not sufficient for the existence of short-term relative arbitrage.
- In particular, there exist $T > 0$ and $\mathbf{\Delta}^d$–valued $\mu(\cdot)$ such that $\mu(\cdot \wedge T)$ is a martingale but (*) holds.
- Such $\mu(\cdot)$ can be an Itô process with a covariation process whose $(d-1)$-st largest eigenvalue is strictly positive (but not bounded away from zero).

Many thanks!

Regular functions

Definition

A continuous function $\boldsymbol{G} : \operatorname{supp}(\mu) \to \mathbb{R}$ is *regular* if

1. there exists a measurable function

$$D\boldsymbol{G} = \big(D_1\boldsymbol{G}, \cdots, D_d\boldsymbol{G}\big)^{\mathsf{T}} : \operatorname{supp}(\mu) \to \mathbb{R}^d$$

   such that the process $\boldsymbol{\vartheta}(\cdot) \in \mathcal{L}(\mu)$ with

$$\boldsymbol{\vartheta}_i(\cdot) = D_i\boldsymbol{G}\big(\mu(\cdot)\big), \qquad i = 1, \cdots, d;$$

2. the continuous, adapted process

$$\Gamma^{\boldsymbol{G}}(\cdot) = \boldsymbol{G}\big(\mu(0)\big) - \boldsymbol{G}\big(\mu(\cdot)\big) + \int_0^{\cdot} \langle \vartheta(t), \mathrm{d}\mu(t) \rangle$$

   has finite variation on compact intervals.

Regular functions

Definition

A continuous function $\boldsymbol{G} : \operatorname{supp}(\mu) \to \mathbb{R}$ is *regular* if

1. there exists a measurable function

$$D\boldsymbol{G} = \left(D_1\boldsymbol{G}, \cdots, D_d\boldsymbol{G}\right)^{\mathsf{T}} : \operatorname{supp}(\mu) \to \mathbb{R}^d$$

   such that the process $\boldsymbol{\vartheta}(\cdot) \in \mathcal{L}(\mu)$ with

$$\boldsymbol{\vartheta}_i(\cdot) = D_i\boldsymbol{G}\big(\mu(\cdot)\big), \qquad i = 1, \cdots, d;$$

2. the continuous, adapted process

$$\Gamma^{\boldsymbol{G}}(\cdot) = \boldsymbol{G}\big(\mu(0)\big) - \boldsymbol{G}\big(\mu(\cdot)\big) + \int_0^{\cdot} \big\langle \vartheta(t), \mathrm{d}\mu(t) \big\rangle$$

   has finite variation on compact intervals.

Lyapunov functions

Definition

We call a regular function $\boldsymbol{G}$ a *Lyapunov function* if the process $\Gamma^{\boldsymbol{G}}(\cdot)$ is non-decreasing.

Remark:

Assume there exists a deflator $Z(\cdot)$ for $\mu(\cdot)$ and $\boldsymbol{G}$ is nonnegative a Lyapunov function for $\mu(\cdot)$. Then

$$Z(\cdot)\boldsymbol{G}(\mu(\cdot)) = Z(\cdot)\left(\boldsymbol{G}(\mu(0)) + \int_0^{\cdot} \sum_{i=1}^{d} D_i\boldsymbol{G}(\mu(t))\mathrm{d}\mu_i(t)\right) - \int_0^{\cdot} \Gamma^{\boldsymbol{G}}(t)\mathrm{d}Z(t) - \int_0^{\cdot} Z(t)\mathrm{d}\Gamma^{\boldsymbol{G}}(t)$$

is a P–local supermartingale, thus also a P–supermartingale as it is nonnegative.

Lyapunov functions

Definition

We call a regular function $\boldsymbol{G}$ a *Lyapunov function* if the process $\Gamma^{\boldsymbol{G}}(\cdot)$ is non-decreasing.

Remark:

Assume there exists a deflator $Z(\cdot)$ for $\mu(\cdot)$ and $\boldsymbol{G}$ is nonnegative a Lyapunov function for $\mu(\cdot)$. Then

$$Z(\cdot)\boldsymbol{G}(\mu(\cdot)) = Z(\cdot)\left(\boldsymbol{G}(\mu(0)) + \int_0^{\cdot}\sum_{i=1}^{d} D_i\boldsymbol{G}(\mu(t))\mathrm{d}\mu_i(t)\right) - \int_0^{\cdot}\Gamma^{\boldsymbol{G}}(t)\mathrm{d}Z(t) - \int_0^{\cdot} Z(t)\mathrm{d}\Gamma^{\boldsymbol{G}}(t)$$

is a P–local supermartingale, thus also a P–supermartingale as it is nonnegative.

Lyapunov functions

Definition

We call a regular function $\boldsymbol{G}$ a *Lyapunov function* if the process $\Gamma^{\boldsymbol{G}}(\cdot)$ is non-decreasing.

Remark:

Assume there exists a deflator $Z(\cdot)$ for $\mu(\cdot)$ and $\boldsymbol{G}$ is nonnegative a Lyapunov function for $\mu(\cdot)$. Then

$$Z(\cdot)\boldsymbol{G}(\mu(\cdot)) = Z(\cdot)\left(\boldsymbol{G}(\mu(0)) + \int_0^{\cdot} \sum_{i=1}^{d} D_i\boldsymbol{G}(\mu(t))\mathrm{d}\mu_i(t)\right) - \int_0^{\cdot} \Gamma^{\boldsymbol{G}}(t)\mathrm{d}Z(t) - \int_0^{\cdot} Z(t)\mathrm{d}\Gamma^{\boldsymbol{G}}(t)$$

is a P–local supermartingale, thus also a P–supermartingale as it is nonnegative.

Concave functions are Lyapunov

Theorem

A continuous function $\boldsymbol{G} : \mathrm{supp}(\mu) \to \mathbb{R}$ *is Lyapunov if it can be extended to a continuous, concave function on*

1. $\boldsymbol{\Delta}_{+}^{d} = \boldsymbol{\Delta}^{d} \cap (0,1)^{d}$ *and*

$$\mathsf{P}\left(\mu(t) \in \boldsymbol{\Delta}_{+}^{d}, \quad \forall\ t \geq 0\right) = 1;$$

2. $\left\{ \left(x_1, \cdots, x_d\right)^{\mathsf{T}} \in \mathbb{R}^d : \sum_{i=1}^{d} x_i = 1 \right\}$ */*
3. $\boldsymbol{\Delta}^{d}$, *and there exists a deflator* $Z(\cdot)$.

Concave functions are Lyapunov

Theorem

A continuous function $\boldsymbol{G} : \mathrm{supp}\,(\mu) \to \mathbb{R}$ is Lyapunov if it can be extended to a continuous, concave function on

1. $\boldsymbol{\Delta}_+^d = \boldsymbol{\Delta}^d \cap (0,1)^d$ *and*

$$\mathsf{P}\left(\mu(t) \in \boldsymbol{\Delta}_+^d\,, \quad \forall\ t \geq 0\right) = 1;$$

2. $\left\{ \left(x_1, \cdots, x_d\right)^{\mathsf{T}} \in \mathbb{R}^d : \sum_{i=1}^d x_i = 1 \right\}$/
3. $\boldsymbol{\Delta}^d$, *and there exists a deflator* $Z(\cdot)$.

Concave functions are Lyapunov

Theorem

A continuous function $\boldsymbol{G} : \mathrm{supp}\,(\mu) \to \mathbb{R}$ *is Lyapunov if it can be extended to a continuous, concave function on*

1. $\boldsymbol{\Delta}^d_+ = \boldsymbol{\Delta}^d \cap (0,1)^d$ *and*

$$\mathsf{P}\left(\mu(t) \in \boldsymbol{\Delta}^d_+\,, \quad \forall\ t \geq 0\right) = 1;$$

2. $\left\{ \left(x_1, \cdots, x_d\right)^{\mathsf{T}} \in \mathbb{R}^d : \sum_{i=1}^d x_i = 1 \right\}$*/*
3. $\boldsymbol{\Delta}^d$, *and there exists a deflator* $Z(\cdot)$.

Outline of the proof

Dellacherie & Meyer:

REMARKS. (a) The same argument would show that, if $X^1, X^2, \ldots, X^n$ are semimartingales and f is a convex function on $\mathbb{R}^n$, the process $f(X_t^1, \ldots, X_t^n)$ is a semimartingale; it is only necessary to know that f is locally Lipschitz, which is true, but rather more delicate[1] than on $\mathbb{R}$.

Outline of the proof

Dellacherie & Meyer:

REMARKS. (a) The same argument would show that, if $X^1, X^2, \ldots, X^n$ are semimartingales and f is a convex function on $\mathbb{R}^n$, the process $f(X_t^1, \ldots, X_t^n)$ is a semimartingale; it is only necessary to know that f is locally Lipschitz, which is true, but rather more delicate[1] than on $\mathbb{R}$.

[1] The proof which appears on p. 370 of the Séminaire de Strasbourg X is pure fantasy.

Outline of the proof

Dellacherie & Meyer:

REMARKS. (a) The same argument would show that, if $X^1, X^2, \ldots, X^n$ are semimartingales and f is a convex function on $\mathbb{R}^n$, the process $f(X^1_t, \ldots, X^n_t)$ is a semimartingale; it is only necessary to know that f is locally Lipschitz, which is true, but rather more delicate[1] than on $\mathbb{R}$.

[1] The proof which appears on p. 370 of the Séminaire de Strasbourg X is pure fantasy.

A paper from 1972:

EVERY CONVEX FUNCTION IS LOCALLY LIPSCHITZ

Wayne State University, Mathematics Department Coffee Room

Outline of the proof

Dellacherie & Meyer:

REMARKS. (a) The same argument would show that, if $X^1, X^2, \ldots, X^n$ are semimartingales and f is a convex function on $\mathbb{R}^n$, the process $f(X_t^1, \ldots, X_t^n)$ is a semimartingale; it is only necessary to know that f is locally Lipschitz, which is true, but rather more delicate[1] than on $\mathbb{R}$.

[1] The proof which appears on p. 370 of the Séminaire de Strasbourg X is pure fantasy.

A paper from 1972:

EVERY CONVEX FUNCTION IS LOCALLY LIPSCHITZ

Wayne State University, Mathematics Department Coffee Room

Also, Rockefeller has a proof.

The dynamics of portfolio log-return

$$
\begin{aligned}
\mathrm{d}\log V^{\pi}(t) &= \frac{\mathrm{d}V^{\pi}(t)}{V^{\pi}(t)} - \frac{1}{2}\sigma_{\pi}^{2}(t)\,\mathrm{d}t \\
&= \sum_{i} \pi_i(t)\,\frac{\mathrm{d}S_i(t)}{S_i(t)} - \frac{1}{2}\sigma_{\pi}^{2}(t)\,\mathrm{d}t \\
&= \sum_{i} \pi_i(t)\Big(\mathrm{d}\log S_i(t) + \frac{1}{2}\sigma_i^{2}(t)\,\mathrm{d}t\Big) - \frac{1}{2}\sigma_{\pi}^{2}(t)\,\mathrm{d}t \\
&= \sum_{i} \pi_i(t)\mathrm{d}\log S_i(t) + \gamma_{\pi}^{*}(t)\,\mathrm{d}t,
\end{aligned}
$$

with $$\gamma_{\pi}^{*}(t) = \frac{1}{2}\Big(\sum_{i} \pi_i(t)\sigma_i^{2}(t) - \sigma_{\pi}^{2}(t)\Big).$$

The excess growth rate

The excess growth rate measures the efficacy of diversification in a portfolio.

$$\begin{aligned}\gamma^*_\pi(t) &= \frac{1}{2}\Big(\sum_i \pi(t)\sigma_i^2(t) - \sigma_\pi^2(t)\Big)\\ &= \frac{1}{2}\Big(\text{weighted average variance} \;-\; \text{portfolio variance}\Big)\\ &\geq 0 \quad \text{for a long-only portfolio.}\end{aligned}$$

The excess growth rate

The excess growth rate measures the efficacy of diversification in a portfolio.

$$\gamma_\pi^*(t) = \frac{1}{2}\Big(\sum_i \pi(t)\sigma_i^2(t) - \sigma_\pi^2(t)\Big)$$
$$= \frac{1}{2}\Big(\text{weighted average variance} \;-\; \text{portfolio variance}\Big)$$
$$\geq 0 \quad \text{for a long-only portfolio.}$$

- The excess growth rate is higher for portfolios of volatile stocks with low correlation. If all else is equal, a higher excess growth rate will increase the long-term performance of a portfolio.

The excess growth rate

The excess growth rate measures the efficacy of diversification in a portfolio.

$$\begin{aligned}\gamma_\pi^*(t) &= \frac{1}{2}\Big(\sum_i \pi(t)\sigma_i^2(t) - \sigma_\pi^2(t)\Big) \\ &= \frac{1}{2}\Big(\text{weighted average variance} \;-\; \text{portfolio variance}\Big) \\ &\geq 0 \quad \text{for a long-only portfolio.}\end{aligned}$$

- The excess growth rate is higher for portfolios of volatile stocks with low correlation. If all else is equal, a higher excess growth rate will increase the long-term performance of a portfolio.
- The formulas assume an implicit rebalancing to the target weights $\pi(t)$ at an infinitesimal time scale reflecting the underlying Brownian motion. Without rebalancing, there's no excess growth.

Decomposition of portfolio log-return

There is a natural decomposition of the log-return of a portfolio into two components. For the interval $[0, T]$,

$$\text{Log-return} \;= \int_0^T \sum_i \pi_i(t)\,\mathrm{d}\log S_i(t) + \int_0^T \gamma_\pi^*(t)\mathrm{d}t.$$

Decomposition of portfolio log-return

There is a natural decomposition of the log-return of a portfolio into two components. For the interval $[0, T]$,

$$\text{Log-return} = \int_0^T \sum_i \pi_i(t)\, \mathrm{d} \log S_i(t) + \int_0^T \gamma_\pi^*(t) \mathrm{d}t.$$

Hence the log return of a portfolio is not only the average of the log returns of its constituents but an additional term appears. (In financial marketing, this phenomenon is sometimes called "volatility harvesting," "volatility pumping," "volatility capture," "rebalancing premium," or "diversification premium")