BSDEs, martingale problems, associated deterministic equations and applications.

Francesco Russo, ENSTA ParisTech

Workshop on "One day on Stochastic Analysis and Applications",

Verona, February 5th 2018

Covers joint work with

Ismail Laachir (Zéliade) and

Adrien Barrasso (ENSTA ParisTech)

Outline

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7. Special case of the Föllmer-Schweizer decomposition.
8. Extensions: the BSDE vs the deterministic problem.

Basic Reference

Ismail Laachir and Francesco Russo.

BSDEs, càdlàg martingale problems and orthogonalization under basis risk.

SIAM Journal on Financial Mathematics, vol. 7, pp. 308-356 (2016)

Related references.

- A. Barrasso and F. Russo.

  Backward Stochastic Differential Equations with no driving martingale, Markov processes and associated Pseudo Partial Differential Equations.

  https://hal.inria.fr/hal-01431559

- A. Barrasso and F. Russo.

  Backward Stochastic Differential Equations with no driving martingale, Markov processes and associated Pseudo Partial Differential Equations. Part II: Decoupled mild solutions and Examples

  https://hal.archives-ouvertes.fr/hal-01505974

- A. Barrasso and F. Russo.
  A note on time-dependent additive functionals .
  Communications on Stochastic Analysis, 11 no 3 (2017), p. 313–334.
- A. Barrasso and F. Russo.
  Martingale driven BSDEs, PDEs and other related deterministic problems.
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Available preprints and publications.
http://uma.ensta.fr/∼russo/

1 General mathematical context

- Interface between “stochastic processes” and “deterministic world”.
- Benchmark situation: bridge between semilinear PDEs and BSDEs.

PDE:

$$\begin{cases} \partial_s u(s,x) + L_s u(s,x) + f(s,x,u(s,x),\sigma\partial_x u(s,x)) = 0 \\ u(T,x) = g(x),\ s \in [0,T], x \in E = \mathbb{R}^d, \end{cases} \quad (1)$$

where L_s is the generator of a diffusion of the type

$$dX_s = \sigma(s,X_s)dW_s + b(t,X_s)ds, X_t = x. \quad (2)$$

BSDE: (2) is coupled with

$$Y_s = g(X_T) + \int_s^T f(s, X_r, Y_r, Z_r)dr - \int_s^T Z_r dW_r. \qquad (3)$$

The link is the following.

1. If u is a classical solution of (1) then

$$Y_s = u(s, X_s), Z_s = \sigma(s, X_s)\nabla u(s, X_s)$$

provide a solution to (3).

2. Viceversa if, given $(t, x) \in [0, T] \times E$ and $X^{t,x}$ is given by (2), $(X^{t,x}, Y^{t,x}, Z^{t,x})$ is a solution to (3), then $u(t, x) := Y_t^{t,x}$ is a *viscosity solution* to (1).

What about $v(t,x) := Z_t^{t,x}$?

- If u is of class $C^{0,1}$ then $v(t,x) = \sigma(t,x)\nabla u(t,x)$.
- What happens in general? Only partial answers even in the Brownian case.
- This talk and the mentioned references discuss some issues related to this problem when W is replaced by a cadlag martingale.

2 Financial Motivations

2.1 Hedging in a complete market

Let $T > 0$, $(\Omega, \mathcal{F}, \mathbb{P})$ a complete probability space with a filtration $(\mathcal{F}_t)_{t\in[0,T]}$, $\mathcal{F}_0$ being the trivial σ-algebra.

- S price of a risky asset.
- B price of a riskless asset.

Complete market.

For any random variable h, there exists a self-financing strategy $(\nu_t)_{t\in[0,T]}$ perfectly replicating h, i.e. a trading strategy that starts from an initial wealth V_0 and re-invests the gain/loss from S on the riskless asset B.

If we suppose that the riskless asset price is constant, this reduces to

$$V_0 + \int_0^T \nu_u dS_u = h.$$

2.2 Hedging in the presence of basis risk

Basis risk.

Risk arising when a derivative product h is *based* on a non-traded or illiquid underlying, but observable, and the replicating (hedging) portfolio is constituted of traded and liquid additional assets which are correlated with the original one.

Example:

- Basket option hedged with a subset of the composing assets.
- Airline companies hedging kerosene exposure with correlated contacts, as crude oil or heating oil.

Consider a pair of processes (X, S) and a contingent claim of the type $h := g(X_T, S_T)$.

- X is a non traded or illiquid, but observable asset.
- S is a traded asset, correlated to X.
- We suppose the riskless asset B to be constant.

Hedging problem: construct a trading strategy on the assets (B, S) in order to replicate the random variable h.

In this case, the market is incomplete: perfect replication with a self-financing strategy is not possible. One should define a risk aversion criterion, for example the following.

- **Utility-based** criterion.
- **Quadratic risk criteria**: *local risk minimization* and *mean-variance minimization*.

2.3 Quadratic hedging: local and global risk minimization.

- Introduced by Föllmer and Sondermann [1985], for S being a (local) martingale. In this case, the unique (local) risk-minimizing strategy is determined by the Kunita-Watanabe (K-W) representation of martingales.
- Extension to the semimartingale case is more delicate, and was handled by Schweizer [1988, 1991]. Its existence is linked to the existence of the so-called Föllmer-Schweizer (F-S) decomposition, a generalization of the (K-W) representation.
- **Global risk minimization.** Again F-S decomposition.

2.4 Föllmer-Schweizer decomposition

Mean-variance hedging is closely related to the so called Föllmer-Schweizer (F-S) decomposition.

Definition 1 *Let* $S = M^S + V^S,\ V_0^S = 0$ *be a special semimartingale. A square integrable random variable* h *admits an F-S decomposition if*

$$h = h_0 + \int_0^T Z_u dS_u + O_T,$$

where $h_0 \in \mathbb{R}, Z \in \Theta$ *and* O *is a square integrable martingale, strongly orthogonal to* M^S*.*

Definition 2 *Let L and N be two $\mathcal{F}_t$-local martingales, with null initial value. L and N are said to be strongly orthogonal if LN is a local martingale.*

Example 3 *If L and N are locally square integrable, then they are strongly orthogonal if and only if $\langle L, N \rangle = 0$.*

2.5 F-S decomposition via a backward SDE

If (h_0, Z, O) is an F-S decomposition, then the process $Y_t := h_0 + \int_0^t Z_u dS_u + O_t$ verifies

$$Y_t := h - \int_t^T Z_u dM_u^S - \int_t^T Z_u dV_u^S - (O_T - O_t),$$

which is a Backward Stochastic Differential Equation, driven by a local martingale, where the final condition $Y_T = h$ is known.
The resolution of the BSDE is a method to determine the F-S decomposition.

3 Backward Stochastic Differential Equations

3.1 BSDEs driven by a Brownian motion

BSDEs were introduced by Pardoux and Peng [1990].
Pioneering work by Bismut [1973].

- Given a pair $(h, \hat{f})$ called *terminal condition* and *driver*.
- One looks for a pair of (adapted) processes (Y, Z), satisfying

$$Y_t = h + \int_t^T \hat{f}(\omega, s, Y_s, Z_s)ds - \int_t^T Z_s dW_s, \qquad (4)$$

and

$$\mathbb{E}\int_0^T |Z_t|^2 dt < \infty.$$

3.2 Existence and uniqueness

- Pardoux and Peng [1990] showed existence and uniqueness when $\hat{f}$ is globally Lipschitz with respect to (y, z) and h being square integrable.
- Conditions on the driver $\hat{f}$ were first relaxed to a monotonicity condition on y, later to a quadratic growth condition and other generalizations, see e.g. Hamadene [1996], Lepeltier and San Martín [1998], Kobylanski [2000], Briand and Hu [2006, 2008].
- Applications to finance: El Karoui et al. [1997].
- Extension to *reflected* BSDEs...

3.3 BSDEs and semi-linear parabolic PDEs

Consider the BSDE

$$Y_s^{t,x} = g(X_T^{t,x}) + \int_s^T f(s, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x})dr - \int_s^T Z_r^{t,x} dW_r, \quad (5)$$

where $\{X_s^{t,x}, t \leq s \leq T\}$ is a solution of the SDE

$$X_s^{t,x} = x + \int_t^s b(r, X_r^{t,x})dr + \int_t^s \sigma(r, X_r^{t,x})dW_r, \ t \leq s \leq T.$$

Link with the semi-linear parabolic PDE.

$$\begin{cases} \partial_t u(t,x) + L_t u(t,x) + f(t,x,u(t,x),\sigma\partial_x u(t,x)) = 0 \\ u(T,x) = g(x),\ t \in [0,T], x \in \mathbb{R}. \end{cases} \tag{6}$$

3.4 From semi-linear parabolic PDEs to BSDEs

Theorem 4 (Pardoux and Peng [1992]) *Let* $u \in C^{1,2}([0,T]\times\mathbb{R},\mathbb{R})$ *be a classical solution of* (6) *such that*

$$|\partial_x u(t,x)| \leq c(1+|x|^q), \text{ for some } c,q>0.$$

Then, $\forall(t,x)$, $(u(s,X_s^{t,x}),(\sigma\partial_x u)(s,X_s^{t,x}))_{s\in[t,T]}$ *is solution of the BSDE* (5).
In particular, under the conditions of well-posedness of the BSDE

$$u(t,x)=Y_t^{t,x}.$$

3.5 From BSDEs to semi-linear parabolic PDEs

Theorem 5 (Pardoux and Peng [1992]) *Let* $(Y_s^{t,x}, Z_s^{t,x})_{s\in[t,T]}$ *be the solution of the BSDE* (5)*, then* $u(t,x) := Y_t^{t,x}$ *is a continuous function and it is a viscosity solution of the PDE* (6)*.*

This representation theorem can be seen as an extension of Feynman-Kac formula.

3.6 Extensions of BSDEs driven by Brownian Motion

- BSDE driven by a Brownian motion and a compensated random measure.
- BSDE driven by a càdlàg martingale.

3.7 BSDEs driven by a càdlàg Martingale

Given a càdlàg (local) martingale M^S and a bounded variation process V^S, one looks for a triplet (Y, Z, O) verifying

$$Y_t = h + \int_t^T \hat{f}(\omega, s, Y_{s-}, Z_s) dV_s^S - \int_t^T Z_s dM_s^S - (O_T - O_t), \quad (7)$$

where O is (local) martingale strongly orthogonal to M^S.

- First contribution by Buckdahn [1993].
- Other contributions, e.g. El Karoui and Huang [1997]. See also Briand et al. [2002], as side-effect of a convergence scheme.
- More recent setting for sufficient conditions for existence and uniqueness for (7) has been given by Carbone et al. [2007].
- BSDEs with partial information driven by càdlàg martingales were investigated by Ceci, Cretarola, Russo in Ceci et al. [2014a,b].

4 Contributions of the work

A forward BSDE, where the forward process solves a *strong martingale problem*. We focus on four tasks.

- Characterize forward-backward SDEs via the solution of a deterministic problem generalizing the classical PDE appearing in the case of Brownian martingales.
- Give applications to the hedging problem in the case of basis risk via the Föllmer-Schweizer decomposition.

- Give explicit expressions when the pair of processes (X, S) is an exponential of additive processes.
- Extensions to the case when the forward process is given in law: strict and generalized solutions of the deterministic problem.

5 Strong Martingale Problem

5.1 Definition

Definition 6 *Let $\mathcal{O}$ be an open set of $\mathbb{R}^2$ and (A_t) be an $\mathcal{F}_t$-adapted b.v. continuous process, such that, a.s. $dA_t \ll d\rho_t$, for some b.v. function ρ, and $\mathcal{A}$ a map*

$$\mathcal{A} : \mathcal{D}(\mathcal{A}) \subset \mathcal{C}([0,T] \times \mathcal{O}, \mathbb{C}) \longrightarrow \mathcal{L}.$$

We say that (X, S) is a solution of the **strong martingale problem** related to $(\mathcal{D}(\mathcal{A}), \mathcal{A}, A)$, if for any $g \in \mathcal{D}(\mathcal{A})$, $(g(t, X_t, S_t))_t$ is a semimartingale such that

$$t \longmapsto g(t, X_t, S_t) - \int_0^t \mathcal{A}(g)(u, X_{u-}, S_{u-}) dA_u$$

is an $\mathcal{F}_t$- local martingale.

Notations 7

- $id : (t,x,s) \longmapsto s$, $s^2 : (t,x,s) \longmapsto s^2$.
- *For any* $y \in \mathcal{C}([0,T] \times \mathcal{O})$, $\widetilde{y} := y \times id$.
- *Suppose that* $id \in \mathcal{D}(\mathcal{A})$. *For* $y \in \mathcal{D}(\mathcal{A})$ *such that* $\widetilde{y} \in \mathcal{D}(\mathcal{A})$, *we set* $\widetilde{\mathcal{A}}(y) := \mathcal{A}(\widetilde{y}) - y\mathcal{A}(id) - id\mathcal{A}(y).$

Proposition 8 *Suppose that $id, s^2 \in \mathcal{D}(\mathcal{A})$. Then S is a special semimartingale with decomposition $M^S + V^S$ given below.*

1. $V_t^S = \int_0^t \mathcal{A}(id)(u, X_{u-}, S_{u-})dA_u$.
2. $\langle M^S \rangle_t = \int_0^t \widetilde{\mathcal{A}}(id)(u, X_{u-}, S_{u-})dA_u$.

Proof.

Item 2. follows from the following more general result.

Lemma 9 *If* $Y_t = y(t, X_t, S_t)$*,* $y, y \times id \in \mathcal{D}(\mathcal{A})$*, then*

$$\langle M^Y, M^S \rangle_t = \int_0^t \tilde{\mathcal{A}}(y)(u, X_{u-}, S_{u-}) dA_u.$$

5.2 Examples

- Diffusion process: the operator $\mathcal{A}$ has the form

$$\begin{aligned} \mathcal{A}(f) &= \partial_t f + b_S \partial_s f + b_X \partial_x f \\ &+ \frac{1}{2}\left\{ |\sigma_S|^2 \partial_{ss} f + |\sigma_X|^2 \partial_{xx} f + 2\langle \sigma_S, \sigma_X \rangle \partial_{sx} f \right\}, \end{aligned}$$

- S is a Markov process, with related Markov semigroup of generator L: the operator $\mathcal{A}$ has the form

$$\mathcal{A}(g)(t,s) = \frac{\partial g}{\partial t}(t,s) + Lg(t,\cdot)(s).$$

5.3 Exponential of additive processes

Definition 10 (Z^1, Z^2) *is said to be an* **additive process** *if* $(Z^1, Z^2)_0 = 0$, (Z^1, Z^2) *is continuous in probability and it has independent increments. The generating function of* (Z^1, Z^2) *is defined by*

$$\exp(\kappa_t(z_1, z_2)) = \mathbb{E}e^{z_1 Z_t^1 + z_2 Z_t^2}, \ \forall (z_1, z_2) \in D,$$

where $D := \{z = (z_1, z_2) \in \mathbb{C}^2 | \quad \mathbb{E}e^{\mathrm{Re}(z_1) Z_T^1 + \mathrm{Re}(z_2) Z_T^2} < \infty\}$.
We denote also, for $(z_1, z_2), (y_1, y_2) \in D/2$

$$\rho_t(z_1, z_2, y_1, y_2) := \kappa_t(z_1 + y_1, z_2 + y_2) - \kappa_t(z_1, z_2) - \kappa_t(y_1, y_2),$$

$$\rho_t^S := \kappa_t(0, 2) - 2\kappa_t(0, 1), \ \textit{if } (0, 1) \in D/2.$$

We always suppose the validity of the following.

Assumption 11 (Basic assumption) $(0,2) \in D$. *This is equivalent to the existence of the second order moment of* $S = e^{Z^2}$.

5.4 First decomposition

We consider two processes $X = \exp(Z^1),\ S = \exp(Z^2)$.

Lemma 12 *Let* $\lambda : [0,T] \times \mathbb{C}^2 \to \mathbb{C}$ *such that, for any* $(z_1, z_2) \in D$*,* $d\lambda(t, z_1, z_2) \ll d\rho_t^S$*. Then for any* $(z_1, z_2) \in D$*,*

$$\begin{aligned} t \quad \mapsto \quad & M_t^\lambda(z_1, z_2) := X_t^{z_1} S_t^{z_2} \lambda(t, z_1, z_2) \\ - \quad & \int_0^t X_{u-}^{z_1} S_{u-}^{z_2} \Big\{ \frac{d\lambda(u, z_1, z_2)}{d\rho_u^S} + \lambda(u, z_1, z_2) \frac{d\kappa_u(z_1, z_2)}{d\rho_u^S} \Big\} \rho_{du}^S, \end{aligned}$$

is a martingale. Moreover, if $(z_1, z_2) \in D/2$ *then* $M^\lambda(z_1, z_2)$ *is a square integrable martingale.*

5.5 Strong Martingale Problem for exponential of additive processes

Theorem 13 *Under some technical assumptions,* (X, S) *is a solution of the strong martingale problem related to* $(\mathcal{D}(\mathcal{A}), \mathcal{A}, \rho^S)$ *where,* $\mathcal{D}(\mathcal{A})$ *is the set of*

$$f : (t, x, s) \mapsto \int_{\mathbb{C}^2} d\Pi(z_1, z_2) x^{z_1} s^{z_2} \lambda(t, z_1, z_2),$$

where Π *is a finite Borel measure on* $\mathbb{C}^2$*,*
$\lambda : [0, T] \times \mathbb{C}^2 \to \mathbb{C}$ *Borel verifying a set of conditions,*

$$
\begin{aligned}
\mathcal{A}(f)(t,x,s) &= \int_{\mathbb{C}^2} d\Pi(z_1,z_2) x^{z_1} s^{z_2} \\
&\quad \left\{ \frac{d\lambda(t,z_1,z_2)}{d\rho_t^S} + \lambda(t,z_1,z_2) \frac{d\kappa_t(z_1,z_2)}{d\rho_t^S} \right\}.
\end{aligned}
$$

6 Deterministic problem related to BSDEs driven by a martingale

6.1 Forward-backward SDE

We consider a pair of $\mathcal{F}_t$-adapted processes (X, S) fulfilling the martingale problem related to $(\mathcal{D}(\mathcal{A}), \mathcal{A}, A)$. We are interested in the BSDE

$$\begin{aligned} Y_t &= g(X_T, S_T) + \int_t^T f(r, X_{r-}, S_{r-}, Y_{r-}, Z_r) dA_r - \int_t^T Z_r dM_r^S \\ &- (O_T - O_t), \end{aligned}$$

1. (Y_t) is $\mathcal{F}_t$-adapted, (Z_t) is $\mathcal{F}_t$-predictable
2. $\int_0^T |Z_s|^2 d\langle M^S \rangle_s < \infty$ a.s.
3. $\int_0^t |f(s, X_{s-}, S_{s-}, Y_{s-}, Z_s)| d\|A\|_s < \infty$ a.s.
4. (O_t) is an $\mathcal{F}_t$-local martingale such that $\langle O, M^S \rangle = 0$ and $O_0 = 0$ a.s.

6.2 Related deterministic analysis

Goal. Look for solutions (Y, Z, O) of the BSDE for which there is a function $y \in \mathcal{D}(\mathcal{A})$ such that $\widetilde{y} = y \times id \in \mathcal{D}(\mathcal{A})$ and a locally bounded Borel function $z : [0, T] \times \mathcal{O} \longrightarrow \mathbb{C}$, such that

$$\begin{aligned} Y_t &= y(t, X_t, S_t), \\ Z_t &= z(t, X_{t-}, S_{t-}), \quad \forall t \in [0, T]. \end{aligned}$$

- When M^S is a Brownian motion, y is a solution of a semilinear PDE.
- General case ?

6.3 Deterministic problem (Pseudo-PDE)

Theorem 14 *Suppose the existence of a function y, such that $y, \widetilde{y} := y \times id$ belong to $\mathcal{D}(\mathcal{A})$, and a Borel locally bounded function z, solving the system*

$$\left\{ \begin{array}{rcl} \mathcal{A}(y)(t,x,s) & = & -f(t,x,s,y(t,x,s),z(t,x,s)) \\ \widetilde{\mathcal{A}}(y)(t,x,s) & = & z(t,x,s)\widetilde{\mathcal{A}}(id)(t,x,s), \end{array} \right.$$

with the terminal condition $y(T,.,.) = g(.,.)$.
Then the triplet (Y,Z,O) defined by

$$Y_t = y(t,X_t,S_t),\ Z_t = z(t,X_{t-},S_{t-})$$

is a solution to the BSDE (8).

7 Special case of the Föllmer-Schweizer decomposition.

7.1 Weak F-S decomposition

Definition 15 *We say that a square integrable $\mathcal{F}_T$-measurable random variable h admits a* **weak F-S decomposition** (h_0, Z, O) *with respect to S if it can be written as*

$$h = h_0 + \int_0^T Z_s dS_s + O_T, \mathbb{P}\text{−a.s.}, \tag{8}$$

where h_0 is an $\mathcal{F}_0$-measurable r.v., Z is a predictable process such that $\int_0^T |Z_s|^2 d\langle M^S\rangle_s < \infty$ a.s., $\int_0^T |Z_s| d\|V^S\|_s < \infty$ a.s. and O is a local martingale such that $\langle O, M^S\rangle = 0$ with $O_0 = 0$.

7.2 Link to BSDEs

Finding a weak F-S decomposition (h_0, Z, O) for some r.v. h is equivalent to provide a solution (Y, Z, O) of the BSDE

$$Y_t = h - \int_t^T Z_s dS_s - (O_T - O_t).$$

The link is given by $Y_0 = h_0$. Here the driver f is linear in z, of the form

$$f(t, x, s, y, z) = -\mathcal{A}(id)(t, x, s)z.$$

$\Rightarrow$ The weak F-S decomposition can be linked to a deterministic problem (Pseudo-PDE).

7.3 Weak Vs True F-S decomposition

Remark 16 *Setting* $h_0 = y(0, X_0, S_0)$*, the triplet* (h_0, Z, O) *is a candidate for a true F-S decomposition. Sufficient conditions for this are the following.*

1. $h = g(X_T, S_T) \in L^2(\Omega)$.
2. $(z(t, X_{t-}, S_{t-}))_t \in \Theta$ *i.e.*
   - $\mathbb{E}\int_0^T |z(t, X_{t-}, S_{t-})|^2 \widetilde{\mathcal{A}}(id)(t, X_{t-}, S_{t-})dA_t < \infty$.
   - $\mathbb{E}\left(\int_0^T |z(t, X_{t-}, S_{t-})| \, \|\mathcal{A}(id)(t, X_{t-}, S_{t-})dA\|_t\right)^2 < \infty$.
3. $\left(y(t, X_t, S_t) - \int_0^t \mathcal{A}(y)(u, X_{u-}, S_{u-})dA_u\right)_t$ *is an* $\mathcal{F}_t$*-square integrable martingale.*

Corollary 17 (Application of the theorem for general BSDEs) *Let y (resp. z): $[0,T] \times \mathcal{O} \to \mathbb{C}$. We suppose the following.*

1. $y, \widetilde{y} := y \times id$ belong to $\mathcal{D}(\mathcal{A})$.
2. $\int_0^T z^2(r, X_{r-}, S_{r-}) \widetilde{\mathcal{A}}(id)(r, X_{r-}, S_{r-}) dA_r < \infty$ a.s.
3. (y, z) solves the problem

$$\begin{cases} \mathcal{A}(y)(t,x,s) & = & \mathcal{A}(id)(t,x,s) z(t,x,s), \\ \widetilde{\mathcal{A}}(y)(t,x,s) & = & \widetilde{\mathcal{A}}(id)(t,x,s) z(t,x,s), \end{cases} \tag{9}$$

with the terminal condition $y(T,.,.) = g(.,.)$.

Then the triplet (Y_0, Z, O), where

$$Y_t = y(t, X_t, S_t),\ Z_t = z(t, X_{t-}, S_{t-}),\ O_t = Y_t - Y_0 - \int_0^t Z_s dS_s,$$

is a weak F-S decomposition of h.

7.4 Application 1: exponential of additive processes

$(X, S) = (e^{Z^1}, e^{Z^2})$ is an exponential of additive processes.

Example 18 *Goal. Use the **Pseudo-PDE** to give explicit expressions of a weak F-S of an $\mathcal{F}_T$-measurable random variable h of the form $h := g(X_T, S_T)$ for a function g of the form*

$$g(x, s) = \int_{\mathbb{C}^2} d\Pi(z_1, z_2) x^{z_1} s^{z_2},$$

where Π is finite Borel complex measure.

Existence and uniqueness.

Proposition 19 *Suppose the validity of the Basic assumption and*

$$\int_0^T \left(\frac{d\kappa_t(0,1)}{d\rho_t^S} \right)^2 d\rho_t^S < \infty.$$

Then any square integrable variable admits a unique true F-S decomposition.

The proof makes use of a general existence and uniqueness theorem by Monat and Stricker [1995].

Idea.

In agreement with the definition of $\mathcal{D}(\mathcal{A})$, we select y of the form

$$y(t,x,s) = \int_{\mathbb{C}^2} d\Pi(z_1,z_2) x^{z_1} s^{z_2} \lambda(t,z_1,z_2),$$

where Π is the same finite complex measure as in the definition of h and $\lambda : [0,T] \times \mathbb{C}^2 \to \mathbb{C}$.

The deterministic equations in the corollary write as

$$
\begin{cases}
\int_{\mathbb{C}^2} d\Pi(z_1, z_2) x^{z_1} s^{z_2} \left\{ \frac{d\lambda(t, z_1, z_2)}{d\rho_t^S} + \lambda(t, z_1, z_2) \frac{d\kappa_t(z_1, z_2)}{d\rho_t^S} \right\} \\
\qquad\qquad = s \frac{d\kappa_t(0,1)}{d\rho_t^S} z(t, x, s) \\
\int_{\mathbb{C}^2} d\Pi(z_1, z_2) \lambda(t, z_1, z_2) x^{z_1} s^{z_2+1} \frac{d\rho_t(z_1, z_2, 0, 1)}{d\rho_t^S} = s^2 z(t, x, s) \\
y(T, \cdot, \cdot) = g.
\end{cases}
$$

Unknown: $\lambda \Rightarrow$ can be determined through the resolution of an ODE in t.

Theorem 20 (Weak F-S decomposition) *Let λ be defined as* $\lambda(t, z_1, z_2) = \exp\left(\int_t^T \eta(z_1, z_2, du)\right), \forall (z_1, z_2) \in D/2,$ *where*

$$\eta(z_1, z_2, t) = \kappa_t(z_1, z_2) - \int_0^t \frac{d\rho_u(z_1, z_2, 0, 1)}{d\rho_u^S} \kappa_{du}(0, 1).$$

Then, under some technical assumptions, (Y_0, Z, O) *is a weak F-S decomposition of* h*, where*

$$Y_t = \int_{\mathbb{C}^2} d\Pi(z_1, z_2) X_t^{z_1} S_t^{z_2} \lambda(t, z_1, z_2),$$

$$Z_t = \int_{\mathbb{C}^2} d\Pi(z_1, z_2) X_{t-}^{z_1} S_{t-}^{z_2 - 1} \lambda(t, z_1, z_2) \gamma_t(z_1, z_2),$$

$$O_t = Y_t - Y_0 - \int_0^t Z_s dS_s \quad \text{and}$$

$$\gamma_t(z_1, z_2) = \frac{d\rho_t(z_1, z_2, 0, 1)}{d\rho_t^S}, \ \forall (z_1, z_2) \in D/2, t \in [0, T],$$

Proposition 21 (True F-S decomposition) *Under slightly stronger assumptions as in Theorem above, the weak F-S decomposition of*

$$h = \int_{\mathbb{C}^2} d\Pi(z_1, z_2) X_T^{z_1} S_T^{z_2}$$

above is a true F-S decomposition.
Moreover, if h is real-valued then the decomposition (h_0, Z, O) is real-valued and it is therefore the unique F-S decomposition.

Example 22 *This statement is a generalization of the results of [Oudjane, Goutte and Russo, 2014] to the case of hedging under basis risk.*

7.5 Application 2: diffusion processes

Let (X, S) be a diffusion process with drift (b_X, b_S) and volatility (σ_X, σ_S).

Assumption 23

- *b_X, b_S, σ_X and σ_S are continuous and globally Lipschitz.*
- *$g : \mathcal{O} \to \mathbb{R}$ is continuous.*

(X,S) solve the strong martingale problem related to $(\mathcal{D}(\mathcal{A}),\mathcal{A},A)$ where $A_t=t$,
$\mathcal{D}(\mathcal{A})=\mathcal{C}^{1,2}([0,T[\times\mathcal{O})\cap\mathcal{C}^1([0,T]\times\mathcal{O})$ and

$$\begin{aligned}\mathcal{A}(y) &= \partial_t y + b_S\partial_s y + b_X\partial_x y \\ &+ \frac{1}{2}\left\{|\sigma_S|^2\partial_{ss}y + |\sigma_X|^2\partial_{xx}y + 2\langle\sigma_S,\sigma_X\rangle\partial_{sx}y\right\}, \\ \widetilde{\mathcal{A}}(y) &= |\sigma_S|^2\partial_s y + \langle\sigma_S,\sigma_X\rangle\partial_x y.\end{aligned}$$

Example 24 ***Goal.*** *characterize the (weak) F-S decomposition of $h := g(X_T,S_T)$.*

Theorem 25 (Weak F-S decomposition) *We suppose the validity of Assumption 23. and that* $|\sigma_S|$ *is always strictly positive. If* (y,z) *is a solution of the system*

$$\begin{cases} \partial_t y + B\partial_x y + \frac{1}{2}\Big(|\sigma_S|^2\partial_{ss} y + |\sigma_X|^2 \partial_{xx} y + 2\langle \sigma_S, \sigma_X\rangle \partial_{sx} y\Big) = 0, \\ y(T,.,.) = g(.,.), \text{ where } B = b_X - b_S \frac{\langle \sigma_S, \sigma_X\rangle}{|\sigma_S|^2}, \\ z = \partial_s y + \frac{\langle \sigma_S, \sigma_X\rangle}{|\sigma_S|^2}\partial_x y, \end{cases} \tag{10}$$

such that $y \in \mathcal{D}(\mathcal{A})$, then (Y_0, Z, O) is a weak F-S decomposition of $g(X_T, S_T)$, where

$$Y_t = y(t, X_t, S_t),\ Z_t = z(t, X_t, S_t),\ O_t = Y_t - Y_0 - \int_0^t Z_s dS_s.$$

Remark 26 1. *Under slightly stronger assumption one can give conditions for the existence of a* **true Föllmer-Schweizer** *decomposition.*

2. *Black-Scholes was treated by Hulley and McWalter [2008].*

8 Extensions: BSDE vs Pseudo-PDEs

- Until now we have essentially shown that a solution to a **Pseudo-PDE** provide solutions to BSDEs driven by cadlag martingales.
- **More problematic is the converse implication.**

  Barrasso and Russo [2017a,b].

Let E be a Polish space. Let $\mathbb{P}^{t,x}$ be a *Markov class* family of probability measures under which the canonical process X on $D([0,T];E)$ solves a martingale problem to $\mathcal{D}(\mathcal{A}), \mathcal{A}, \rho)$. Let us denote $M^S := M_s^{id,t} := S_s - x - \int_t^s \mathcal{A}(id)(S_r)d\rho(r)$. We consider $BSDE(f, g(S_T), M^S)$, i.e.

$$\begin{aligned} Y_s &= g(S_T) + \int_s^T f(r, S_{r-}, Y_{r-}, Z_r)d\rho_r - \int_s^T Z_r dM_r^S \\ &- (O_T - O_s), s \in [t,T], \end{aligned} \tag{11}$$

under $\mathbb{P}^{t,x}$.

Let us suppose the following.

- $id \in \mathcal{D}(\mathcal{A})$.
- $\langle M^S \rangle$ is absolutely continuous with respect to ρ.
- Let us suppose suitable growth condition on g and Lipschitz on f.

“Theorems”

- Then (11) admits a unique solution $(Y^{t,x}, Z^{t,x}, O^{t,x})$ in some suitable spaces.
- There is a “unique” couple (y, z) of Borel functions such that $y(t,x) = Y_t^{t,x}$, and $Z_s^{t,x} = z(s, X_s)$ a.s. under $\mathbb{P}^{t,x}$.
- There is a *unique* so called decoupled mild solution of Pseudo-PDE(f, g). It is given by (y, z).
- Extensions to the *Path-dependent case.*

Thank you for your attention!

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