Rough volatility from an affine point of view

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  - Hawkes processes and rough volatility models in mathematical finance;
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  - Development of stochastic models which have this feature: Gatheral, Jaisson & Rosenbaum (14); Guennoun, Jacquier & Roome (14); Bayer, Friz & Gatheral (15); El Euch & Rosenbaum (16); Bennedsen, Lunde, Pakkanen (16).
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- Ambit processes, with the subclass of Brownian semistationary processes, have been considered by Barndorff-Nielsen, Benth & Veraart to model turbulence and energy prices.

Properties and related challenges

- Important features and challenges of stochastic Volterra processes are
  - non-Markovianity,
  - non-semimartingality,

  leading to path dependence of conditional expectations, memory effects and particular trajectorial properties such as the ubiquitous roughness.
- Several techniques known in the Markovian world can be carried over to the Volterra world, often by replacing classical derivatives by convolutional expressions.
- This holds in particular true for certain techniques known for affine and polynomial processes, e.g. the affine transform formula with fractional Riccati equations holds true (see ER(16) and ALP(17)).

Motivating examples: Hawkes process and rough Heston model

- A (one-dimensional) Hawkes process N is a process that jumps by 1 with intensity

$$\lambda_t = \lambda_0 + \int_0^t \varphi(t-s)\theta(s)ds + \int_0^t \varphi(t-s)dN_s.$$

with a locally integrable kernel φ and a deterministic function θ.

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- The rough Heston model consists of a log-price Y and a instantaneous variance process X such that

$$Y_t = Y_0 - \frac{1}{2}\int_0^t X_s ds + \int_0^t \sqrt{X}_s dB_{s,2},$$

$$X_t = X_0 + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\kappa(\theta - X_s)ds + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\sqrt{X}_s dB_{s,1},$$

where $\alpha = H + \frac{1}{2} \in (\frac{1}{2}, 1)$ and B_1 and B_2 are correlated Brownian motions.

Goal of today's talk

Goal of this work

1. Unifying Markovian framework for general stochastic Volterra processes via transport stochastic partial differential equations.
2. (Numerical) approximations of Volterra processes via finite dimensional Markov processes
3. Special case of affine characteristics - affine transform formula.

Stochastic Volterra equations - Setting

- State space $E = C \times \mathbb{R}^k \subseteq \mathbb{R}^d$, with C a closed proper convex cone.
- The components of an E-valued process Z are denoted by $Z = (X, Y)$.
- Consider an E-valued stochastic Volterra equation with càglàd paths:

$$Z_t = Z_0 + \int_0^t K(t-s)\theta(s)ds - \int_0^t K(t-s)BZ_s ds + \int_0^t K(t-s)dM_s,$$

where

  - K denotes a matrix valued kernel in $L^2_{\mathrm{loc}}(\mathbb{R}_+, \mathbb{R}^{d\times d})$ of block diagonal form $K = \begin{pmatrix} K_X & 0 \\ 0 & K_Y \end{pmatrix}$
  - M denotes an $\mathbb{R}^d$-valued martingale such that each component is in $\mathcal{H}^2_{\mathrm{loc}}$ and $\langle M, M \rangle_{t,ij} = \int_0^t c_{ij}(Z_s)ds + \int_0^t \int_{\mathbb{R}^d} \xi_i \xi_j F(Z_s, d\xi)ds$ for some function $c : \mathbb{R}^d \to \mathbb{S}^d_+$ and some Borel kernel F from $\mathbb{R}^d$ into $\mathbb{R}^d$ s.t. $\int_0^t c_{ii}(z)ds + \int_0^t \int_{\mathbb{R}^d} \xi_i^2 F(z, d\xi)ds \leq c(1 + \|z\|^2)$ for every i.
  - $\theta \in L^1_{\mathrm{loc}}(\mathbb{R}_+, \mathbb{R}^d)$
  - $B \in \mathbb{R}^{d\times d}$

Resolvents - Notation

- The resolvent of K is the kernel $R \in L^2_{\text{loc}}(\mathbb{R}_+, \mathbb{R}^{d\times d})$ that satisfies

$$K * R = R * K = K - R.$$

- We shall also consider the resolvent of the first kind whenever it exists. This is an $\mathbb{R}^{d\times d}$-valued measure on $\mathbb{R}_+$ of locally bounded variation such that

$$K * L = L * K = I_d.$$

- We write R^B for the resolvent of KB and $N := K - R^B * K$.

Examples

- $K(t) = 1, \quad R(t) = \exp(-t), \quad L(dt) = \delta_0(dt);$
- $K(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)}, \quad R(t) = t^{\alpha-1} E_{\alpha,\alpha}(t^{-\alpha}), \quad L(dt) = \frac{t^{-\alpha}}{\Gamma(1-\alpha)} dt$

Completely monotone kernels

- In order to guarantee the existence of L we shall often assume that

Assumption

$$K(t) \in \mathbb{S}_d^{++} \textit{ for some } t > 0 \textit{ and completely monotone on } (0, \infty),$$

i.e. K is infinitely often differentiable and

$$(-1)^j K^{(j)}(t) \in \mathbb{S}_d^{+}, \quad t \in (0, \infty), \quad j = 0, 1, 2, \ldots,$$

where $K^{(j)}$ denotes the j^{th} derivative.

- If K is completely monotone then by an extension of Bernstein's theorem to the matrix case it is the Laplace transform of an $\mathbb{S}_d^+$-valued measure α on $\mathbb{R}_+$, i.e.

$$K(t) = \int_0^\infty e^{-xt} \alpha(dx),$$

with $\alpha(A) \in \mathbb{S}_d^+$ for every bounded Borel set in $\mathbb{R}_+$.

"Forward processes" and their state space

- First goal: Find a Markovian structure behind these Volterra equations
- Let Z be a solution of the above stochastic Volterra equation. Define the "forward" process
$$W_t(x) := \mathbb{E}[Z_{t+x}|\mathcal{F}_t]$$
and note that $W_t(0) = Z_t$.

Theorem (C., Teichmann (2017))

Assume that K is completely monotone and that $\theta_X(x)dx \succeq -L_X(dx)X_0$. Then the function valued process $(x \mapsto W_t(x))_{t\geq 0}$ takes values in

$$\mathcal{E} = \{f : \mathbb{R}_+ \to E \,|\, f(x) = (I_d - \int_0^x R^B(s)ds)Z_0 + \int_0^x N(x-s)\theta(s)ds$$
$$\text{with } Z_0 = (X_0, Y_0) \in E,\ \theta \in L^1_{loc}(\mathbb{R}_+, \mathbb{R}^d) \text{ s.t. } \theta_X(x)dx \succeq -L_X(dx)X_0\}.$$

"Forward processes" and associated SPDEs

Proposition (C., Teichmann (2017))

Assume that K is completely monotone and that $\theta_X(x)dx \succeq -L_X(dx)X_0$. Then the forward curve process $(W_t)_{t\geq 0}$ is a time-homogenous Markov process taking values in $\mathcal{E}$-valued continuous curves and satisfies the SPDE

$$\begin{aligned} dW_t(x) &= \frac{d}{dx}W_t(x)dt + N(x)dM_t, \\ W_0(x) &= \left(I_d - \int_0^x R^B(s)ds\right)Z_0 + \int_0^x N(x-s)\theta(s)ds, \end{aligned} \qquad (*)$$

in the following mild pointwise ("Walsh") sense

$$\begin{aligned} W_t(x) &= S_t W_0(x) + \int_0^t S_{t-s}N(x)dM_s \\ &= W_0(x+t) + \int_0^t N(x+t-s)dM_s, \end{aligned}$$

where $(S_t)_{t\geq 0}$ denotes the shift semigroup.

Equivalence of existence of solutions for the Volterra equation and the SPDE

Corollary (C., Teichmann (2017))

Assume that K is completely monotone. TFAE:

1. *The SPDE (*) with M being a $\mathcal{H}^2_{loc}$ martingale starting at 0 such that*

$$\langle M, M\rangle_{t,ij} = \int_0^t c_{ij}(W_s(0))ds + \int_0^t \int_{\mathbb{R}^d} \xi_i \xi_j F(W_s(0), d\xi)ds$$

   for some function $c : \mathbb{R}^d \to \mathbb{S}^d_+$ and some Borel kernel F from $\mathbb{R}^d$ into $\mathbb{R}^d$ admits an $\mathcal{E}$-valued mild pointwise solution for any initial value in $\mathcal{E}$.

2. *The Volterra equation admits an E-valued solution for all $Z_0 \in E$ and $\theta \in L^1_{loc}(\mathbb{R}^+, \mathbb{R}^d)$ satisfying*

$$\theta_X(x)dx \succeq -L_X(dx)X_0.$$

Remarks

- Existence of stochastic Volterra equations for enough curves θ can be translated to existence of the above SPDE and vice versa.
- Embedding the state space $\mathcal{E}$ in an appropriate Hilbert space allows to consider solutions of the above SPDE in the usual mild sense.
- Note that by construction via $\mathbb{E}[Z_{t+x}|\mathcal{F}_t]$, $W_{X,t}(x) \in C$ for all $x \geq 0$. This necessarily implies that whenever $W_{X,t}(x) \in \partial C$ for some $x > 0$ that $W_{X,t}(0) \in \partial C$ because this enters in the volatility. This is one of the invariance properties the state space $\mathcal{E}$ satisfies.

Towards approximations via intermediate processes

- Second goal: Approximate Volterra equation and the corresponding SPDE via a sequence of intermediate processes to obtain existence and numerical methods (compare Frederico & Tankov "Finite-dimensional representations for controlled diffusions with delay")
- An intermediate process V is defined on $E \times D$ with $E = C \times \mathbb{R}^k \subset \mathbb{R}^d$ and $D \subseteq \mathbb{R}^n$.
- $\pi : \mathbb{R}^{d+n} \to \mathbb{R}^d$ is the canonical projection and $Z_t := \pi V_t$.
- An intermediate process V takes the following form

$$V_t = V_0 + \int_0^t b(s)ds + \int_0^t AV_s ds - \int_0^t \begin{pmatrix} BZ_s \\ 0 \end{pmatrix} ds + \begin{pmatrix} M_t \\ 0 \end{pmatrix},$$

where $b \in L^1_{\mathrm{loc}}(\mathbb{R}_+, \mathbb{R}^{d+n})$, $A \in \mathbb{R}^{(d+n)\times(d+n)}$, $B \in \mathbb{R}^{d\times d}$, M a H^2_{loc} martingale as above.

Volterra form of the projected intermediate processes

Denote $P_t = \exp(At)$. Then by the variation of constants formula the E-valued process Z

$$Z_t = \pi P_t V_0 + \int_0^t \pi P_{t-s} b(s) ds - \int_0^t \pi P_{t-s} \pi B Z_s ds + \int_0^t \pi P_{t-s} \pi dM_s$$

is (nearly) of Volterra form as defined above, with kernel

$$K(t) = \pi P_t \pi = \pi(e^{At})\pi$$

however with time varying initial value and term $\int_0^t \pi P_{t-s} b(s) ds$ which is a priori not of form $\int_0^t K(t-s)\theta(s) ds$.

Volterra form of the projected intermediate processes

Lemma

Let $V_0 = (Z_0, U_0)'$ and decompose the matrix A according to the product structure $E \times D$ as follows

$$A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}.$$

Let $b = (\theta - \alpha Z_0 - \beta U_0, -\gamma Z_0 - \delta U_0)'$ for some $\theta \in L^1_{loc}(\mathbb{R}_+, \mathbb{R}^d)$. Then Z is exactly of Volterra form (without time varying intial value).

Approximating sequence of intermediate processes and their forward processes

- Consider a sequence of matrices $A^n \in \mathbb{R}^{(d+n)\times(d+n)}$ growing in the dimension and σ_i^n for $i \in \{1, \ldots, d\}$. Then we can consider the following types of kernels

$$K(t) = \lim_{n\to\infty} K^n(t) = \lim_{n\to\infty} \pi(e^{A^n t})\pi \operatorname{diag}(\sigma_1^n, \ldots, \sigma_d^n) \quad \text{in } L^2_{\text{loc}}(\mathbb{R}_+, \mathbb{R}^{d\times d}).$$

- In particular, completely monotone kernels can be generated and we can go beyond diagonal kernels.
- Consider the corresponding sequence of intermediate processes $(V_t^n)_{t\geq 0}$ and their forward processes $(W_t^n)_{t\geq 0}$.
- Depending on the characteristics of the martingale, we either obtain strong or weak convergence of $(W_t^n)_{t\geq 0}$. In the subsequent proposition we have

$$M_t^n = \int_0^t \sqrt{c(W^n(0))}dB_s + \int_0^t \int_{\mathbb{R}^d} \xi(\mu^J(d\xi, ds) - F(W^n(0), d\xi)ds),$$

and $N^n = K^n - (R^B)^n * K^n$.

Strong convergence

Proposition (C., Teichmann (2017))

Let $(W_t^n)_{t\geq 0}$ be a sequence of forward processes which satisfy the SPDE

$$dW_t^n(x) = \frac{d}{dx} W_t^n(x)dt + N^n(x)dM_t^n, \quad W_0^n \in \mathcal{E}^n$$

in the mild pointwise ("Walsh") sense. Assume that $W_0^n \to W_0$ uniformly on compacts and assume $\|N^n - N\|_{L^2_{loc}} \to 0$. Furthermore we assume that for every $T \geq 0$

$$\mathbb{E}\left[\left\|\int_0^T N^m(x+T-s)^2 d[M^n - M^m, M^n - M^m]_s\right\|\right] \leq C\mathbb{E}\left[\int_0^T N^m(x+T-s)^2 \|W_s^n(0) - W_s^m(0)\|^2 ds\right].$$

Then $\mathbb{E}\left[\sup_{t\leq T}\|W_t^n(x) - W_t^m(x)\|^2\right] \to 0$ as $n, m \to \infty$, for every $x \geq 0$ and $T \geq 0$.

Affine characteristics - Setting

- Third goal: Analysis of the case with affine characteristics
- We here assume that the martingale is affine, i.e. the predictable quadratic variation of M given by

$$\langle M, M \rangle_{t,ij} = \int_0^t c_{ij}(W_s(0))ds + \int \xi_i \xi_j F(W_s(0), d\xi)ds$$

has linear characteristics, i.e.

$$c(z) = \sum_{i=1}^{d} c_i z_i, \quad c_i \in \mathbb{S}^d \text{ s.t. } \quad c(z) \in \mathbb{S}_+^d \text{on } E,$$
$$F(z, d\xi) = \langle z, \nu(d\xi) \rangle,$$

where ν is a d-dimensional vector valued (signed) measure on $\mathbb{R}^d$ with bounded support such that $F(x, d\xi)$ is a Lévy measure for every $x \in E$.

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  where ν is a d-dimensional vector valued (signed) measure on $\mathbb{R}^d$ with bounded support such that $F(x, d\xi)$ is a Lévy measure for every $x \in E$.
- The above notion of an "affine martingale" corresponds to an affine process in usual sense with linear (instead of affine) characteristics. This is for ease of notation, the "true" affine case can be treated analogously.

Existence assumption

- $\mathcal{M}$ denotes the space of d-dimensional signed measures on $\mathbb{R}_+$.

Assumption A

Let K be a completely monotone kernel. Assume that the SPDE (*) with M an affine martingale admits a probabilistically weak and PDE weak solution with values in $\mathcal{E}$, i.e. for every $\mu \in \mathcal{M} \cap \mathcal{D}(-\frac{d}{dx})$

$$\int_0^\infty \langle W_t(x), \mu(dx) \rangle = \int_0^\infty \langle W_0(x), \mu(dx) \rangle - \int_0^t \int_0^\infty \langle W_s(x), \frac{d}{dx}\mu(dx) \rangle ds$$
$$+ \int_0^t \left\langle \int_0^\infty N(x)'\mu(dx), \sqrt{c(W_s(0))} dB_s \right\rangle$$
$$+ \int_0^t \left\langle \int_0^\infty N(x)'\mu(dx), \int_{\mathbb{R}^d} \xi(\mu(d\xi, dt) - F(W_s(0), d\xi)ds) \right\rangle,$$

holds in a probabilistically weak sense. We here express M deliberately as $M_t = \int_0^\cdot \sqrt{c(W_s(0))} dB_s + \int_0^t \int_{\mathbb{R}^d} \xi(\mu(d\xi, ds) - F(W_s(0), d\xi)ds$.

Towards the affine transform formula

- This solution concept does not need any infinte dimensional stochastic integration theory, since the involved martingales and integrands are all finite dimensional.
- Consider the following "dual space" of $\mathcal{E}$

$$\mathcal{U} = \{\mu \in \mathcal{M} + \mathrm{i}\mathcal{M} \mid \exp\left(\int_0^\infty \langle f(x)\mu(dx)\rangle\right) < \infty \text{ for all } f \in \mathcal{E}\}.$$

- For a function $V : \mathcal{M} \to \mathbb{R}$, $\partial_\mu, V(\mu)(x)$ denotes the derivative in direction δ_x, i.e.

$$\partial_\mu V(\mu)(x) = \lim_{\varepsilon \to 0} \frac{V(\mu + \varepsilon\delta_x) - V(\mu)}{\varepsilon}.$$

Affine transform formula

Theorem (C., Teichmann (2017))

Under assumption A, $(W_t(x))_{t\geq 0}$ is an affine process on $\mathcal{E}$ in the sense that for all $\mu_0 \in \mathcal{U} \cap \mathcal{D}(-\frac{d}{dx})$

$$\mathbb{E}_w \left[\exp\left(\int_0^\infty \langle W_t(x), \mu_0(dx) \rangle \right) \right]$$

solves the following transport equation

$$\partial_t f(t,\mu_0) = \int_0^\infty \langle \partial_{\mu_0} f(t,\mu_0)(x), R(\mu_0)(dx) \rangle, \quad f(0,\mu_0) = e^{\int_0^\infty \langle w(x), \mu_0(dx) \rangle},$$

where the $\mathbb{C}^d$-valued measure $R(\mu_0)$ is given by

$$R(\mu_0)(dx) = -\frac{d}{dx}\mu_0(dx) + \delta_0(dx)\mathcal{R}\left(\int_0^\infty N(x)'\mu_0(dx)\right)$$

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$$R(\mu_0)(dx) = -\frac{d}{dx} \mu_0(dx) + \delta_0(dx) \mathcal{R}(\int_0^\infty N(x)' \mu_0(dx))$$

with $\mathcal{R} : \mathbb{C}^d \to \mathbb{C}^d$,
$\mathcal{R}(u) = \frac{1}{2} (\langle u, c_1 u \rangle, \ldots, \langle u, c_d u \rangle)' + \int_{\mathbb{R}^d} (\exp(\langle u, \xi \rangle) - 1 - \langle u, \xi \rangle) \nu(d\xi).$

Affine transform formula

Theorem (cont.)

If the following measure-valued PDE $\partial_t \mu_t(dx) = R(\mu_t)(dx)$ *admits a unique global mild solution, i.e.*

$$\mu_t(dx) = \mu_0(dx - t) + \int_0^t \delta_0(dx - t + s)\mathcal{R}\Big(\int_0^\infty N(x)'\mu_s(dx)\Big)ds.$$

Then the unique solution of the transport equation is given by $\exp(\int_0^\infty \langle w(x), \mu_t(dx)\rangle)$ *so that*

$$\mathbb{E}_w\left[\exp\Big(\int_0^\infty \langle W_t(x), \mu_0(dx)\rangle\Big)\right] = \exp\Big(\int_0^\infty \langle w(x), \mu_t(dx)\rangle\Big).$$

Connection to Riccati Volterra equations

- Approximate $\mu_0(dx) = u\delta_0(dx)$ with $u \in E^*$.
- Define $\widetilde{\psi}(t) := \int_0^\infty N(x)' \mu_t(dx)$, where μ_t is the solution of the above Riccati PDE with intial condition μ_0.
- Then the above result translates to

$$\mathbb{E}_w \left[\exp(\langle W_t(0), u \rangle)\right] = \exp\left(\langle w(t), u \rangle + \int_0^t \langle w(s), \mathcal{R}(\widetilde{\psi}(t-s)) ds \rangle \right).$$

- Since $W_t(0)$ satisfies the stochastic affine Volterra equation

$$W_t(0) = V + \int_0^t K(t-s)h(s)ds - \int_0^t K(t-s)BW_s(0) + \int K(t-s)dM_s$$

this give the Fourier-Laplace transform of affine Volterra processes in terms of their forward process.

Connection to Riccati Volterra equations

- In the diffusion case this is the same representation as in Abi Jaber, Larsson, Pulido (2017), since

$$\widetilde{\psi}(t) := \int_0^\infty N(x)' \mu_t(dx)$$

satisfies the generalized Riccati Volterra equation

$$\widetilde{\psi}(t) = N(t)'u + \int_0^t N(t-s)' \mathcal{R}(\widetilde{\psi}(s)) ds,$$

where

$$\mathcal{R}(u) = \frac{1}{2} \left(\langle u, c_1 u \rangle, \ldots, \langle u, c_d u \rangle \right)' + \int_{\mathbb{R}^d} \left(\exp(\langle u, \xi \rangle) - 1 - \langle u, \xi \rangle \right) \nu(d\xi).$$

Relation to the Riccati equations of the intermediate process

- Consider an affine intermediate process $(V_t)_{t\geq 0}$ with $B=0$ (for simplicity) and kernel $K(t)=N(t)=\pi e^{At}\pi$ and .
- Then assumptions in the above theorem on the primal side (existence of $W_t(x)=\mathbb{E}[Z_{t+x}|\mathcal{F}_t]$) and on the dual side (Riccati equations) are clearly satisfied.
- In particular we have the following representation of the Riccati PDE in terms of the characteristic exponent $\psi(t,u)$ of the intermediate process V

$$\mu_t(dx)=\mu_0(dx-t)+\int_0^t \delta_0(dx-t+s)\mathcal{R}\left(\pi\psi\left(s,\int_0^\infty e^{A'x}\begin{pmatrix}\mu_0(dx)\\0\end{pmatrix}\right)\right)ds$$

such that

$$\widetilde{\psi}(t)=\int_0^\infty \pi e^{A'x}\pi\mu_t(dx)=\pi\psi\left(t,\int_0^\infty e^{A'x}\begin{pmatrix}\mu_0(dx)\\0\end{pmatrix}\right)$$

Towards existence & uniqueness via kernel approximation

For kernels of the type

$$K(x) = \lim_{n\to\infty} \pi(e^{A^n x})\pi \operatorname{diag}(\sigma_1^n, \ldots, \sigma_m^n) \quad \text{in } L^2_{\text{loc}}(\mathbb{R}_+, \mathbb{R}^{m\times m}).$$

we obtain

- (probabilistically) weak and (PDE) weak solutions of the SPDE $(W_t)_{t\geq 0}$;
- existence of global solutions to the Riccati equations;
- uniqueness in law;
- numerical approximations by standard affine processes.

Conclusion and Outlook

- Conclusion
  - We provide a generic Markovian structure for general Volterra equations via the forward process $(W_t)_{t \geq 0}$.
  - For a big class of kernels obtained as limits of scaled entries of matrix exponentials, we obtain numerical approximations via intermediate processes.
  - In the case of affine characteristics, we have an affine transform formula for the process $(W_t)_{t \geq 0}$.
- Outlook
  - Exploit Markovian structure also in other respects (polynomial processes, etc.)

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Thank you for your attention!