ON WONG-ZAKAI APPROXIMATIONS FOR A FAMILY OF STOCHASTIC DIFFERENTIAL EQUATIONS INTERPOLATING BETWEEN THE ITÔ AND STRATONOVICH INTERPRETATIONS

Alberto Lanconelli

Università degli Studi di Bari Aldo Moro

1 Itô and Stratonovich SDEs

A *stochastic differential equation* driven by a standard one dimensional Brownian motion $\{B_t\}_{t\geq 0}$ is an equation of the type

$$dX_t = b(t, X_t)dt + \sigma(t, X_t)dB_t, \quad X_0 = x$$

whose rigorous interpretation is the integral equation

$$X_t = x + \int_0^t b(s, X_s)ds + \int_0^t \sigma(s, X_s)dB_s, \text{ for all } t \geq 0.$$

Here $b, \sigma : [0, +\infty[\times\mathbb{R} \to \mathbb{R}$ are measurable functions and $x \in \mathbb{R}$.

Stochastic *differential* equations are indeed *integral* equations. This is due to the fact that

- the function $t \mapsto B_t$ is almost surely *nowhere non differentiable*

Moreover,

- the function $t \mapsto B_t$ ia almost surely of *unbounded variation*

hence, the term

$$\int_0^t \sigma(s, X_s) dB_s$$

can not be defined through a Riemann-Stiltjes integral.

There are different ways to define an integral with respect to Brownian motions. Each of such definition will produce a *different* interpretation of the SDE

$$X_t = x + \int_0^t b(s, X_s) ds + \int_0^t \sigma(s, X_s) dB_s.$$

1. **Itô integral (1944)**

$$\int_a^b Z_s dB_s := \lim_{n \to +\infty} \sum_{j=1}^n Z_{t_{j-1}} \cdot (B_{t_j} - B_{t_{j-1}}) \quad \text{in } \mathcal{L}^2(\Omega, \mathcal{F}, \mathcal{P})$$

   Nice properties:

$$E\Big[\int_a^b Z_s dB_s\Big] = 0 \quad \text{and} \quad t \mapsto \int_a^t Z_s dB_s \quad \text{is a martingale.}$$

2. **Stratonovich integral (1966)**

$$\int_a^b Z_s \circ dB_s := \lim_{n \to +\infty} \sum_{j=1}^n Z_{\frac{t_{j-1}+t_j}{2}} \cdot (B_{t_j} - B_{t_{j-1}}) \quad \text{in probability}$$

   Nice property:

$$f(B_t) = f(B_0) + \int_0^t f'(B_s) \circ dB_s$$

Itô-Stratonovich relationship

It is known that the Stratonovich SDE

$$dX_t = b(t, X_t)dt + \sigma(t, X_t) \circ dB_t, \quad X_0 = x$$

is equivalent to the Itô SDE

$$dX_t = \Big(b(t, X_t) + \frac{1}{2}\sigma(t, X_t)\partial_x \sigma(t, X_t)\Big)dt + \sigma(t, X_t)dB_t, \quad X_0 = x.$$

2 Wong-Zakai-type theorems

For $\varepsilon > 0$ let $\{B_t^\varepsilon\}_{t\geq 0}$ be a smooth approximation of the Brownian motion $\{B_t\}_{t\geq 0}$, i.e.

- $t \mapsto B_t^\varepsilon$ is continuously differentiable
- B_t^ε converges a.s. to B_t uniformly (on compacts) in t

Take for instance

$$B_t^\varepsilon := \int_0^{+\infty} \varphi_\varepsilon(t-s)B_s ds$$

where φ_ε is a mollifier. Now, consider the *random* ordinary differential equation

$$\frac{dX_t^\varepsilon}{dt} = b(t, X_t^\varepsilon) + \sigma(t, X_t^\varepsilon) \cdot \frac{dB_t^\varepsilon}{dt}, \quad X_0^\varepsilon = x. \tag{2.1}$$

Theorem 2.1 (Wong-Zakai (1965))
The solution $\{X_t^\varepsilon\}_{t\geq 0}$ of equation (2.1) converges as $\varepsilon \to 0$ to the solution of the Stratonovich SDE

$$dX_t = b(t, X_t)dt + \sigma(t, X_t) \circ dB_t, \quad X_0 = x.$$

More precisely, for any $T > 0$ one has

$$\lim_{\varepsilon\to 0} E\Big[\sup_{0\leq t\leq T} |X_t^\varepsilon - X_t|^2\Big] = 0.$$

Example 2.2
Let $b(t,x) = 0$ and $\sigma(t,x) = x$. Then, equation (2.1) reads

$$\frac{dX_t^\varepsilon}{dt} = X_t^\varepsilon \cdot \frac{dB_t^\varepsilon}{dt}, \quad X_0^\varepsilon = x$$

whose solution is

$$X_t^\varepsilon = x \exp\{B_t^\varepsilon\}.$$

Therefore,

$$\lim_{\varepsilon\to 0} X_t^\varepsilon = x\exp\{B_t\} \qquad \text{since } B_t^\varepsilon \to B_t \text{ almost surely.}$$

Using Itô formula one can see that $\{x\exp\{B_t\}\}_{t\geq 0}$ solves the Itô SDE

$$dX_t = \frac{1}{2}X_t dt + X_t dB_t, \quad X_0 = x$$

which corresponds to the Stratonovich SDE

$$dX_t = X_t \circ dB_t, \quad X_0 = x$$

Which is the *correct* random ordinary differential equation converging to the Itô SDE

$$dX_t = b(t, X_t)dt + \sigma(t, X_t)dB_t, \quad X_0 = x \quad ?$$

Recall that the Stratonovich SDE

$$dX_t = b(t, X_t)dt + \sigma(t, X_t) \circ dB_t, \quad X_0 = x$$

is equivalent to the Itô SDE

$$dX_t = \Big[b(t, X_t) + \frac{1}{2}\sigma(t, X_t)\partial_x\sigma(t, X_t)\Big]dt + \sigma(t, X_t)dB_t, \quad X_0 = x.$$

Therefore, one may consider

$$\frac{dZ_t^\varepsilon}{dt} = \Big[b(t, Z_t^\varepsilon) - \frac{1}{2}\sigma(t, Z_t^\varepsilon)\partial_x\sigma(t, Z_t^\varepsilon)\Big] + \sigma(t, Z_t^\varepsilon) \cdot \frac{dB_t^\varepsilon}{dt}, \quad Z_0^\varepsilon = x.$$

By the Wong-Zakai theorem Z_t^ε converges to the solution of

$$dZ_t = \Big[b(t, Z_t) - \frac{1}{2}\sigma(t, Z_t)\partial_x\sigma(t, Z_t)\Big]dt + \sigma(t, Z_t) \circ dB_t, \quad Z_0 = x$$

which is equivalent to

$$dZ_t = b(t, Z_t)dt + \sigma(t, Z_t)dB_t, \quad Z_0 = x.$$

This procedure is not completely satisfactory. In fact, first of all some probabilistic properties of the exact solution are lost in the approximation

Example 2.3
If we wish to approximate

$$dX_t = X_t dB_t, \quad X_0 = x$$

according to previous procedure, then we should consider the random ODE

$$\frac{dX_t^\varepsilon}{dt} = -\frac{1}{2}X_t^\varepsilon + X_t^\varepsilon \cdot \frac{dB_t^\varepsilon}{dt}, \quad X_0^\varepsilon = x.$$

whose solution is

$$X_t^\varepsilon = x \exp\Big\{B_t^\varepsilon - \frac{1}{2}t\Big\}.$$

Therefore, on one hand we have

$$E[X_t] = x \quad \textit{for all } t \geq 0$$

while on the other

$$E[X_t^\varepsilon] \neq x \textit{ unless } E[(B_t^\varepsilon)^2] = t \textit{ for all } \varepsilon > 0.$$

Moreover, the Stratonovich interpretation is not always available.

Remark 2.4
Consider the stochastic partial differential equation

$$du(t,x) = [\partial_{xx}u(t,x) + b(u(t,x))]dt + \sigma(u(t,x))dB_{tx} \tag{2.2}$$

where $\{B_{tx}\}_{(t,x)\in[0,T]\times[0,1]}$ *is a two-parameter Brownian motion.*
For this equation there is no Stratonovich interpretation since the candidate correction term *to add to the Itô equation would be infinite.*
Hairer and Pardoux (2015) proved that the solution of the random PDE

$$\begin{aligned} \partial_t u^\varepsilon(t,x) &= \partial_{xx}u^\varepsilon(t,x) + \tilde{b}(u^\varepsilon(t,x)) - \frac{c_1}{\varepsilon}\sigma(u^\varepsilon(t,x))\sigma'(u^\varepsilon(t,x)) \\ &\quad + \sigma(u^\varepsilon(t,x)) \cdot \partial_{tx}B^\varepsilon_{tx}, \end{aligned} \tag{2.3}$$

where

$$\tilde{b}(u) := b(u) - c_2(b'(u))^3 b(u) - c_3 b''(u)b'(u)(b(u))^2$$

with constants c_1*,* c_2 *and* c_3 *which depends on the mollifier, converges in probability to the Itô's type SPDE*

$$du(t,x) = [\partial_{xx}u(t,x) + b(u(t,x))]dt + \sigma(u(t,x))dB_{tx}.$$

2.1 Wick product and Hu-Øksendal theorem

Definition 2.5
For $f \in L^2(\mathbb{R}_+)$ *let*

$$\mathcal{E}(f) := \exp\left\{\int_0^{+\infty} f(t)dB_t - \frac{1}{2}\int_0^{+\infty} f^2(t)dt\right\}.$$

For $f, g \in L^2(\mathbb{R}_+)$ *we define the* Wick product *of* $\mathcal{E}(f)$ *and* $\mathcal{E}(g)$ *as*

$$\mathcal{E}(f) \diamond \mathcal{E}(g) := \mathcal{E}(f+g).$$

Extend by linearity as

$$\begin{aligned} \Big(\sum_{j=1}^n \alpha_j \mathcal{E}(f_j)\Big) \diamond \Big(\sum_{i=1}^m \beta_i \mathcal{E}(g_i)\Big) &:= \sum_{j=1}^n \sum_{i=1}^m \alpha_j \beta_i \mathcal{E}(f_j) \diamond \mathcal{E}(g_i) \\ &= \sum_{j=1}^n \sum_{i=1}^m \alpha_j \beta_i \mathcal{E}(f_j + g_i). \end{aligned}$$

Now take $X, Y \in \mathcal{L}^p(\Omega)$ for some $p \geq 1$. Then,

$$X \diamond Y := \lim_{n\to+\infty} \Big(\sum_{j=1}^{n} \alpha_j \mathcal{E}(f_j)\Big) \diamond \Big(\sum_{i=1}^{n} \beta_i \mathcal{E}(g_i)\Big) \qquad \textit{(in a proper weak topology)}$$

where

$$\Big\|\sum_{j=1}^{n} \alpha_j \mathcal{E}(f_j) - X\Big\|_p \to 0 \qquad \textit{and} \qquad \Big\|\sum_{j=1}^{n} \beta_j \mathcal{E}(f_j) - Y\Big\|_p \to 0.$$

Example 2.6

$$\begin{aligned} B_t \diamond B_s &= B_t \cdot B_s - t \wedge s \\ \varphi(B_t) \diamond \psi(B_s) &= \varphi(B_t) \cdot \psi(B_s) - \varphi'(B_t) \cdot \psi'(B_s)(t \wedge s) \\ &\quad -\frac{1}{2}\varphi''(B_t) \cdot \psi''(B_s)(t \wedge s)^2 + \ldots \\ \varphi(B_t) \diamond \mathcal{E}(f) &= \varphi\Big(B_t - \int_0^t f(s)ds\Big) \cdot \mathcal{E}(f) \\ X \diamond (B_t - B_s) &= X \cdot (B_t - B_s) \quad \textit{if } X \textit{ is } \mathcal{F}_s^B\textit{-measurable} \end{aligned}$$

Note that from the last identity we can write in the definition of Itô integral

$$\int_a^b Z_s dB_s = \lim_{n\to+\infty} \sum_{j=1}^{n} Z_{t_{j-1}} \cdot (B_{t_j} - B_{t_{j-1}}) = \lim_{n\to+\infty} \sum_{j=1}^{n} Z_{t_{j-1}} \diamond (B_{t_j} - B_{t_{j-1}})$$

Nice properties of the *Wick product*

- $E[X \diamond Y] = E[X] \cdot E[Y]$
- if $\{X_t\}_{t\in[0,T]}$ and $\{Y_t\}_{t\in[t,T]}$ are $\{\mathcal{F}_t^B\}_{t\in[0,T]}$-martingales, then $\{X_t \diamond Y_t\}_{t\in[0,T]}$ is an $\{\mathcal{F}_t^B\}_{t\in[0,T]}$-martingale.

The Wick product is a *Gaussian convolution*:

$$E[(X \diamond Y)\mathcal{E}(f)] = E[X\mathcal{E}(f)] \cdot E[Y\mathcal{E}(f)] \quad \text{for all } f \in L^2(\mathbb{R}_+)$$

(recall that $\int_{\mathbb{R}} (f \star g)(x) e^{i\lambda x} dx = \int_{\mathbb{R}} f(x) e^{i\lambda x} dx \cdot \int_{\mathbb{R}} g(x) e^{i\lambda x} dx$ for all $\lambda \in \mathbb{R}$).

Theorem 2.7 (Hu-Øksendal (1995))
As before, for $\varepsilon > 0$ let $\{B_t^\varepsilon\}_{t\geq 0}$ be a smooth approximation of the Brownian motion $\{B_t\}_{t\geq 0}$. Now, consider the following modified *random differential equation*

$$\frac{dY_t^\varepsilon}{dt} = b(t, Y_t^\varepsilon) + \sigma(t, Y_t^\varepsilon) \diamond \frac{dB_t^\varepsilon}{dt}, \quad Y_0^\varepsilon = x. \tag{2.4}$$

Assume that $\sigma(t,x) = \hat{\sigma}(t)x$ where $\hat{\sigma}$ is a bounded function of t. Then, the solution $\{Y_t^\varepsilon\}_{t\geq 0}$ of equation (2.4) converges as $\varepsilon \to 0$ to the solution of the Itô's type SDE

$$dY_t = b(t, Y_t)dt + \sigma(t, Y_t)dB_t, \quad Y_0 = x.$$

More precisely, for any $T > 0$ one has

$$\lim_{\varepsilon\to 0} E\Big[\sup_{0\leq t\leq T} |Y_t^\varepsilon - Y_t|^2\Big] = 0.$$

Remark 2.8
We observe that the assumption $\sigma(t,x) = \hat{\sigma}(t)x$ in the previous theorem is used to prove the existence of a unique solution for equation (2.4).

Example 2.9
Let $b(t,x) = 0$ and $\sigma(t,x) = x$. Then, equation (2.4) reads

$$\frac{dY_t^\varepsilon}{dt} = Y_t^\varepsilon \diamond \frac{dB_t^\varepsilon}{dt}, \quad X_0^\varepsilon = x.$$

whose solution is

$$Y_t^\varepsilon = x \exp\Big\{B_t^\varepsilon - \frac{E[|B_t^\varepsilon|^2]}{2}\Big\}.$$

Therefore,

$$\lim_{\varepsilon\to 0} Y_t^\varepsilon = x \exp\Big\{B_t - \frac{t}{2}\Big\} \quad \text{since} \quad B_t^\varepsilon \to B_t \text{ almost surely.}$$

Using Itô formula one can verify that the limit solves the Itô SDE

$$dY_t = Y_t dB_t, \quad X_0 = x.$$

3 The interpolation

3.1 The α-product

For $\lambda \geq 0$ and $X = \sum_{n\geq 0} I_n(h_n) \in \mathcal{L}^2(\Omega)$ we define the operator

$$\Gamma(\lambda)X = \Gamma(\lambda)\sum_{n\geq 0} I_n(h_n) := \sum_{n\geq 0} I_n(\lambda^n h_n).$$

We observe that

$$\Gamma(0)X = E[X] \qquad \Gamma(1)X = X \qquad \Gamma(\lambda)\Gamma(\theta) = \Gamma(\lambda\cdot\theta)$$

Moreover, since

$$\|\Gamma(\lambda)X\|^2_{\mathcal{L}^2(\Omega)} = \sum_{n\geq 0} \lambda^{2n} n! |h_n|^2_{L^2(\mathbb{R}^n_+)}$$

we see that the operator $\Gamma(\lambda)$ is unbounded for $\lambda > 1$, while it coincides with the Ornstein-Uhlenbeck semigroup when $\lambda \in]0,1]$. In fact, writing $\lambda = e^{-\tau}$ for some $\tau \geq 0$ we get

$$(\Gamma(e^{-\tau})X)(\omega) = (P_\tau X)(\omega) := \int_\Omega X\left(e^{-\tau}\omega + \sqrt{1-e^{-2\tau}}\tilde{\omega}\right) d\mu(\tilde{\omega}), \quad \omega \in \Omega$$

which is a contraction on $\mathcal{L}^p(\Omega)$ for any $p \geq 1$ (and hyper-contractive for $p > 1$).

Definition 3.1
Let $\alpha > 0$. The α-product of X and Y, *denoted by $X \circ_\alpha Y$, is defined as*

$$X \circ_\alpha Y := \Gamma\left(\frac{1}{\sqrt{\alpha}}\right) (\Gamma(\sqrt{\alpha})X \cdot \Gamma(\sqrt{\alpha})Y). \tag{3.1}$$

It is straightforward to check that the α-product is commutative, associative and distributive with respect to the sum. While it is trivial to observe that for $\alpha = 1$ the α-product coincides with the pointwise product $X \cdot Y$, it is not obvious to guess what happens to the α-product if we let α tend to zero.

Proposition 3.2
Let X and Y be smooth random variables. Then,

$$\lim_{\alpha \to 0^+} X \circ_\alpha Y = X \diamond Y.$$

Therefore, the α-product interpolates between the *pointwise* product, when $\alpha = 1$ and the *Wick* product, when $\alpha = 0$. The previous proposition also explains why the Wick product allows to multiply generalized random variables.

We observe that for $h \in L^2(\mathbb{R}_+)$ we have

$$X \circ_\alpha \int_0^{+\infty} h(s) dB_s = X \cdot \int_0^{+\infty} h(s) dB_s + (\alpha - 1) D_h X$$

where $D_h X$ denotes the *Malliavin derivative* of X in the direction $\int_0^{\cdot} h(s)ds$.

3.2 Wong-Zakai type theorem

We consider

$$B_t^\varepsilon := \int_0^T K_\varepsilon(t,s)dB_s, \quad t \in [0,T]$$

(recall that $B_t = \int_0^T 1_{[0,t]}(s)dB_s$) where for any $\varepsilon > 0$ the function $K_\varepsilon : [0,T]^2 \to \mathbb{R}$ is such that

- the function $t \mapsto K_\varepsilon(t,s)$ belongs to $C^1([0,T])$ for almost all $s \in [0,T]$;
- the functions $s \mapsto K_\varepsilon(t,s)$ and $s \mapsto \partial_t K_\varepsilon(t,s)$ belong to $L^2([0,T])$ for all $t \in [0,T]$.

Moreover, we assume that

$$\lim_{\varepsilon \to 0^+} \sup_{t \in [0,T]} |K_\varepsilon(t,\cdot) - 1_{[0,t]}(\cdot)|_{L^2([0,T])} = 0 \tag{3.2}$$

and

$$M := \sup_{\varepsilon > 0} \sup_{t \in [0,T]} |K_\varepsilon(t,\cdot)|_{L^2([0,T])} < +\infty.$$

Then, the previous assumptions on K imply that $\{B_t^\varepsilon\}_{t\in[0,T]}$ is a continuously differentiable Gaussian process and that B_t^ε converges to B_t in $\mathcal{L}^2(\Omega)$ uniformly with respect to $t \in [0,T]$. In fact, condition (3.2) is equivalent to

$$\lim_{\varepsilon \to 0^+} \sup_{t \in [0,T]} \|B_t^\varepsilon - B_t\|_{\mathcal{L}^2(\Omega)} = 0.$$

Theorem 3.3 (The approximated equation)
Fix $\alpha \in]0,1]$ and let $b : [0,T] \times \mathbb{R} \to \mathbb{R}$ be such that

$$|b(t,x) - b(t,y)| \;\leq\; L|x-y| \quad \text{for all } x,y \in \mathbb{R}, t \in [0,T]$$

for some positive constants L. Let $\hat{\sigma} : [0,T] \to \mathbb{R}$ be a bounded deterministic function. Then, for any $\varepsilon > 0$ the equation

$$\frac{dZ_t^\varepsilon}{dt} = b(t, Z_t^\varepsilon) + \hat{\sigma}(t) Z_t^\varepsilon \circ_\alpha \frac{dB_t^\varepsilon}{dt}, \quad Z_0^\varepsilon = x$$

admits a unique solution belonging to $\mathcal{L}^p(\Omega)$ for any $p \geq 1$.

Let $\alpha \in [0,1]$ and denote

$$\int_a^b Z_t d^\alpha B_t := \int_a^b Z_t dB_t + \frac{\alpha}{2}\langle Z, B\rangle_t,$$

where the integral in the right hand side is of Itô's type and $\langle Z, B\rangle_t$ denotes the covariation between $\{Z_t\}_{t\in[a,b]}$ and $\{B_t\}_{t\in[a,b]}$. This corresponds to

$$\int_a^b Z_t d^\alpha B_t = \lim_{n\to+\infty} \sum_{j=1}^n Z_{(1-\frac{\alpha}{2})t_{j-1}+\frac{\alpha}{2}t_j} \cdot (B_{t_j} - B_{t_{j-1}}) \quad \text{in probability.}$$

Theorem 3.4 (Convergence to the exact solution)
Let $\{Z_t\}_{t\in[0,T]}$ be the unique solution of the SDE

$$dZ_t = b(t, Z_t)dt + \hat{\sigma}(t)Z_t d^\alpha B_t, \quad t \in]0,T] \qquad Z_0 = x$$

and for any $\varepsilon > 0$ let $\{Z_t^\varepsilon\}_{t\in[0,T]}$ be the unique solution of

$$\frac{dZ_t^\varepsilon}{dt} = b(t, Z_t^\varepsilon) + \hat{\sigma}(t)Z_t^\varepsilon \circ_\alpha \frac{dB_t^\varepsilon}{dt}, \quad Z_0^\varepsilon = x.$$

Then, for any $p \geq 1$ there exists a positive constant C (depending on α, p, $|x|$, T, L and M) such that for any q greater than p

$$\sup_{t\in[0,T]} \|Z_t^\varepsilon - Z_t\|_p \leq C \cdot \mathcal{S}_q\left(\sqrt{2} \sup_{t\in[0,T]} |K_\varepsilon(t,\cdot) - 1_{[0,t]}(\cdot)|\right)$$

where

$$\mathcal{S}_q(\lambda) := \lambda \exp\left\{q\lambda^2\right\} + \exp\{\lambda^2/2\} - 1, \quad \lambda \in \mathbb{R}$$

Theorem 3.5 (Itô-Stratonovich relationship)
For any $\varepsilon > 0$ let $\{Y_t^\varepsilon\}_{t\in[0,T]}$ be the unique solution of

$$\frac{dY_t^\varepsilon}{dt} = b(t, Y_t^\varepsilon) + \sigma(t)Y_t^\varepsilon \diamond \frac{dB_t^\varepsilon}{dt}, \quad Y_0^\varepsilon = x.$$

Then, for any $t \in [0,T]$ we have

$$Y_t^\varepsilon = \mathcal{T}_{-K_\varepsilon(t,\cdot)} S_t^\varepsilon, \tag{3.3}$$

where $\{S_t^\varepsilon\}_{t\in[0,T]}$ is the unique solution of

$$\frac{dS_t^\varepsilon}{dt} = b(t, S_t^\varepsilon) + \frac{1}{2}\frac{d|K_\varepsilon(t,\cdot)|^2}{dt} \cdot S_t^\varepsilon + \sigma(t)S_t^\varepsilon \cdot \frac{dB_t^\varepsilon}{dt}, \quad S_0^\varepsilon = x.$$