Alpha-CIR Model in Sovereign Interest Rate Modelling

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Motivation

- Current sovereign bond markets in the Euro zone :
  - ◇ persistency of low interest rates
  - ◇ significant fluctuations at local extent.

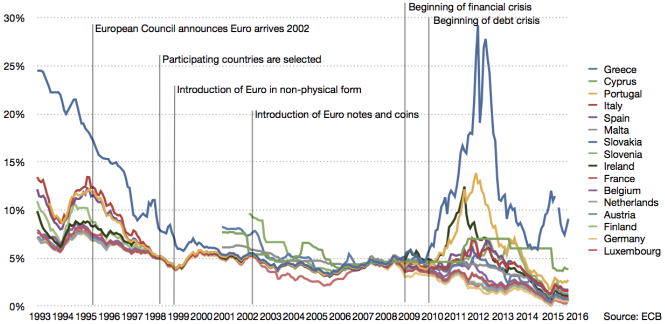

FIGURE – Long term interest rates of Euro area countries.

Modelling approaches

- Large fluctuations in financial data motivate the introduction of jumps in the interest rate dynamics : Eberlein & Raible (1999), Filipović, Tappe & Teichmann (2010)...
- Hawkes process to model the "self-exciting" feature : Aït-Sahalia & Jacod (2009), Errais, Giesecke & Goldberg (2010), Dassios & Zhao (2011), Rambaldi, Pennesi & Lillo (2014), and Jaisson & Rosenbaum (2015)...
- Difficulty : jump presence v.s. trend of low interest rate

Plan of our work and main result

- Objective : a new model of interest rate (α-CIR model) for these seemingly puzzling phenomena in a unified and parsimonious framework.
- Jump diffusion model as natural extension of the CIR model, using the α-stable branching processes
  - CIR model is the particular case with continuous path
- Integral representation to highlight the branching property :
  - limit of Hawkes processes : clustering and self-exciting properties ;
  - link with CBI processes : exponential affine structure for bond price, Duffie, Filipović & Schachermayer (2001)
- The bond price decreases with the parameter α, which allows to respond to the low interest rate behavior.
  - surprising result :
  - when α decreases, the tails are heavier. The "risks" become larger but the bond price increases....
  - comparaison with CIR model with "Poisson" α-stable jumps.
  - different effect if jumps are in branching mechanism or immigration rate, very low interest rates in α-CIR model.

The α-CIR model setup : Integral representation (Dawson-Li)

Integral form by using the random fields

$$
\begin{aligned}
r_t = r_0 &+ \int_0^t a\,(b - r_s)\,ds + \sigma \int_0^t \int_0^{r_s} W(ds, du) \\
&+ \sigma_Z \int_0^t \int_0^{r_{s-}} \int_{\mathbb{R}+} \zeta \widetilde{N}(ds, du, d\zeta),
\end{aligned} \tag{1}
$$

- $W(ds, du)$: white noise on $\mathbb{R}_+^2$ with intensity $dsdu$,
- $\widetilde{N}(ds, du, d\zeta)$: compensated Poisson random measure on $\mathbb{R}_+^3$ with intensity $dsdu\mu(d\zeta)$,
- $\mu(d\zeta)$ is a Lévy measure satisfying $\int_0^\infty (\zeta \wedge \zeta^2)\mu(d\zeta) < \infty$.

Random fields for interest rate modelling : Kennedy (1994), Albeverio, Lytvynov & Mahnig (2004).

The α-CIR model setup

We consider α-CIR$(a, b, \sigma, \sigma_Z, \alpha)$ model for the short interest rate

$$r_t = r_0 + \int_0^t a\,(b - r_s)\,ds + \sigma \int_0^t \sqrt{r_s} dB_s + \sigma_Z \int_0^t r_{s-}^{1/\alpha} dZ_s \tag{2}$$

- $B = (B_t, t \geq 0)$ a Browinan motion
- $Z = (Z_t, t \geq 0)$ a spectrally positive α-stable compensate Lévy process with parameter $\alpha \in (1, 2]$ with
$$\mathbb{E}\left[e^{-qZ_t}\right] = \exp\left\{-\frac{tq^\alpha}{\cos(\pi\alpha/2)}\right\}, \quad q \geq 0.$$
- B and Z are independent

Z_t follows the α-stable distribution $S_\alpha(t^{1/\alpha}, 1, 0)$ with scale parameter $t^{1/\alpha}$, skewness parameter 1 and zero drift.

Simulation of processes Z and r with different α

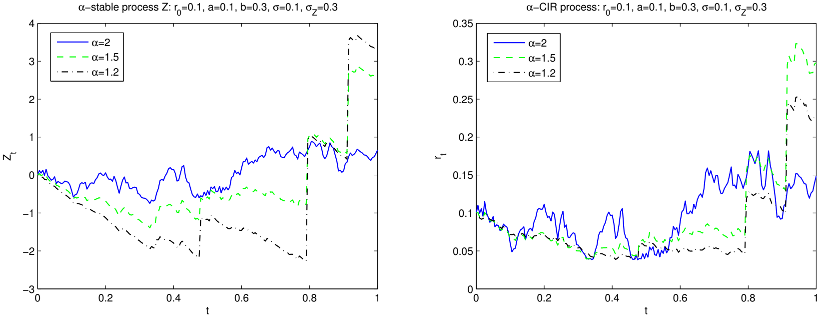

FIGURE – Three parameters of α : 2 (blue), 1.5 (green) and 1.2 (black)

Similar properties with CIR model

Boundary condition :

The point 0 is an inaccessible boundary if and only if $2ab \geq \sigma^2$. In particular, a pure jump α-CIR process with $ab > 0$ never reaches 0 since $\sigma = 0$.

Branching property :

r can be decomposed as $r = r^{(1)} + r^{(2)}$ where for $i = 1, 2$, $r^{(i)}$ is an α-CIR$(a, b^{(i)}, \sigma, \sigma_Z, \alpha)$ process such that $r_0 = r_0^{(1)} + r_0^{(2)}$ and $b = b^{(1)} + b^{(2)}$.
This property is a direct consequence of

- linearity of integrals
- homogeneity of measures

Equivalence of two representations

We choose the Lévy measure to be

$$\mu(d\zeta) = -\frac{1_{\{\zeta>0\}}d\zeta}{\cos(\pi\alpha/2)\Gamma(-\alpha)\zeta^{1+\alpha}}, \quad 1 < \alpha < 2, \tag{3}$$

Then the root representation (2) and the integral representation (1) are equivalent in the following sense :

- The solutions of the two equations have the same probability law.
- On an extended probability space, they are equal almost surely.

Link to Hawkes process

- When $\sigma = 0$ and $\mu(d\zeta) = \delta_1(dz)$, then r is given by

$$r_t = r_0 + abt - \int_0^t (a + \sigma_Z) r_s ds + \sigma_Z \int_0^t \int_0^{r_{s-}} N(ds, du) \tag{4}$$

  which is the intensity of Hawkes process $\int_0^t \int_0^{r_{s-}} N(ds, du)$, N being the Poisson random measure with intensity $dsdu$.

- Consider a sequence $\left\{ r_t^{(n)}, t \geq 0 \right\}$ defined by (4) with parameters $(a/n, nb, \sigma_Z)$. Then

$$r_{nt}^{(n)}/n \xrightarrow{\mathcal{L}} Y_t \quad \text{in } D(\mathbb{R}_+),$$

  where $D(\mathbb{R}_+)$ is the Skorokhod space of càdlàg processes and

$$Y_t = \int_0^t a(b - Y_s) ds + \sigma_Z \int_0^t \int_0^{Y_s} W(ds, du).$$

- Jaisson and Rosenbaum (2015) : nearly unstable Hawkes process converges, after suitable scaling, to a CIR process.

Locally equivalent CIR process with jumps

- Consider the α-CIR process with initial value r_0 and introduce

$$\begin{aligned}\lambda_t = r_0 + \int_0^t a\,(b-\lambda_s)\,ds + \sigma \int_0^t \int_0^{\lambda_s} W(ds,du) \\ + \sigma_Z \int_0^t \int_0^{r_0} \int_{\mathbb{R}+} \zeta \widetilde{N}(ds,du,d\zeta)\end{aligned} \tag{5}$$

  where the processes W and $\widetilde{N}$ are (almost) the same as in (2).
- the above CIR process with jumps can be written as

$$d\lambda_t = r_0 + a\,(b-\lambda_t)\,dt + \sigma\sqrt{\lambda_t}dB_t + \sigma_Z\sqrt[\alpha]{r_0}dZ_t,$$

- The implicit negative drifts lead to a linear decay for λ_t while an exponential decay for r_t : when σ_Z increases, the decreasing drift plays a more important role in α-CIR than in equivalent CIR with jumps.

Comparison between α-CIR and CIR with α-stable jumps (continued)

- Separating small and large jumps in CIR with jumps, we get

$$\lambda_t = r_0 + \int_0^t a\Big(b - \frac{\sigma_Z r_0 \Theta(\alpha, y)}{a} - \lambda_s\Big)ds + \sigma \int_0^t \int_0^{\lambda_s} W(ds, du)$$
$$+ \sigma_Z \int_0^t \int_0^{r_0} \int_0^y \zeta \widetilde{N}(ds, du, d\zeta) + \sigma_Z \int_0^t \int_0^{r_0} \int_y^\infty \zeta N(ds, du, d\zeta)$$

where

$$\Theta(\alpha, y) = \frac{2}{\pi} \alpha \Gamma(\alpha - 1) \frac{\sin(\pi\alpha/2)}{y^{\alpha-1}}.$$

- In a similar way, the α-CIR process can be written as

$$r_t = r_0 + \int_0^t \widetilde{a}(\alpha, y)\Big(\widetilde{b}(\alpha, y) - r_s\Big)ds + \sigma \int_0^t \int_0^{r_s} W(ds, du)$$
$$+ \sigma_Z \int_0^t \int_0^{r_{s-}} \int_0^y \zeta \widetilde{N}(ds, du, d\zeta) + \sigma_Z \int_0^t \int_0^{r_{s-}} \int_y^\infty \zeta N(ds, du, d\zeta)$$

where

$$\widetilde{a}(\alpha, y) = a + \sigma_Z \Theta(\alpha, y), \quad \widetilde{b}(\alpha, y) = \frac{ab}{a + \sigma_Z \Theta(\alpha, y)}$$

Continuous state branching process with immigration (CBI)

CBI (Kawazu & Watanabe 1971) of branching mechanism $\Psi(\cdot)$ and immigration rate $\Phi(\cdot)$: Markov process X with state space $\mathbb{R}_+$ verifying

$$\mathbb{E}_x\left[e^{-pX_t}\right] = \exp\left[-xv(t,p) - \int_0^t \Phi\big(v(s,p)\big)ds\right],$$

where $v : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$ satisfies

$$\frac{\partial v(t,p)}{\partial t} = -\Psi(v(t,p)), \quad v(0,p) = p$$

and Ψ and Φ are functions on $\mathbb{R}_+$ given by

$$\Psi(q) = \beta q + \frac{1}{2}\sigma^2 q^2 + \int_0^\infty (e^{-qu} - 1 + qu)\pi(du),$$
$$\Phi(q) = \gamma q + \int_0^\infty (1 - e^{-qu})\nu(du),$$

with $\sigma, \gamma \geq 0$, $\beta \in \mathbb{R}$ and π, ν being two Lévy measures such that $\int_0^\infty (u \wedge u^2)\pi(du) < \infty$ and $\int_0^\infty (1 \wedge u)\nu(du) < \infty$.

Link with the CBI processes

Let r be an α-CIR $(a, b, \sigma, \sigma_Z, \alpha)$ process. Then r is a CBI with

$$\text{branching mechanism} : \Psi(q) = aq + \frac{\sigma^2}{2} q^2 - \frac{\sigma_Z^\alpha}{\cos(\pi\alpha/2)} q^\alpha \tag{6}$$

$$\text{immigration rate} : \Phi(q) = abq. \tag{7}$$

Consequences :

- As $t \to +\infty$, r_t has a limite distribution r_∞, given by

$$\mathbb{E}[e^{-pr_\infty}] = \exp\left\{-\int_0^p \frac{\Phi(q)}{\Psi(q)} dq\right\}, \quad p \geq 0.$$

- Laplace transform

$$\mathbb{E}\left[e^{-\xi r_t - p\int_0^t r_s ds}\right] = \exp\Big(-r_0 v(t,\xi,p) - \int_0^t \Phi\big(v(s,\xi,p)\big)ds\Big),$$

$$\text{with } \partial_t v(t,\xi,p) = -\Psi(v(t,\xi,p)) + p, \quad v(0,\xi,p) = \xi.$$

- Let $r^{(\alpha)}$ be α-CIR$(a, b, \sigma, \sigma_Z, \alpha)$ process, $\alpha \in (1, 2]$. Then $r^{(\alpha)} \xrightarrow{\mathcal{L}} r^{(2)}$ in $D(\mathbb{R}_+)$ as $\alpha \to 2$.

Application to bond pricing

For simplicity, we assume that the short rate r is given by an α-CIR$(a, b, \sigma, \sigma_Z, \mu, \alpha)$ model under $\mathbb{Q}$.

- Zero-coupon bond price :

$$B(t,T) = \exp\Big(- r_t v(T-t) - ab \int_0^{T-t} v(s) ds \Big),$$

  where $v(\cdot)$ is given by

$$\frac{\partial v(t)}{\partial t} = 1 - \Psi(v(t)), \quad v(0) = 0,$$

  with $\Psi(q) = aq + \frac{\sigma^2}{2} q^2 - \frac{\sigma_Z^\alpha}{\cos(\pi\alpha/2)} q^\alpha$.

- We have

$$v(t) = f^{-1}(t) \text{ where } f(t) = \int_0^t \frac{dx}{1 - \Psi(x)} \tag{8}$$

Proposition

The function $v(\cdot)$ is increasing with respect to $\alpha \in (1,2]$. In particular, the bond price $B(0,T)$ is decreasing with respect to α.

- α characterizes the tail fatness : when α decreases, it is more likely to take values far away from median and have large jumps.
- Generalized Blumenthal-Getoor index (e.g. Aït-Sahalia and Jacod, 2009) $\inf\{\beta > 0 : \sum_{0\le s\le T} \Delta r_s^\beta < \infty,\ a.s.\}$ equals $\inf\{\beta > 0 : \int_0^T r_s ds \int_0^1 u^\beta \mu(du) < \infty,\ a.s.\} = \alpha.$
- The above proposition shows that the α-CIR model is suitable to describe the phenomenon of low interest rate trend with jumps.

Bond behaviour of α-CIR model

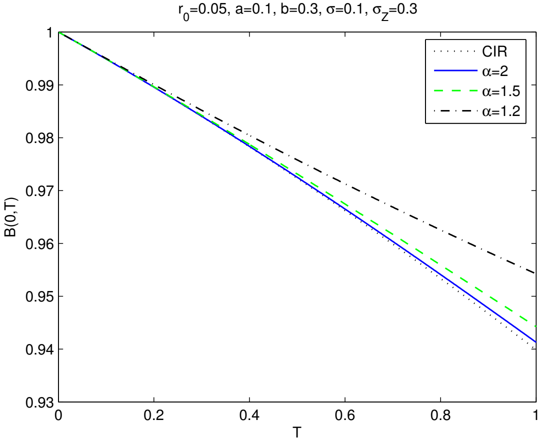

FIGURE – Bond price is decreasing w.r.t. α, curve CIR corresponds to $\sigma_Z = 0$

Yield curve

$$Y(t,\theta) = -\frac{1}{\theta}\ln B(t,\theta) = r_t\frac{v(\theta)}{\theta} + \frac{ab}{\theta}\int_0^{\theta} v(s)ds$$

According to Keller-Ressel and Steiner (08), define $x_0 > 0$ as the unique solution of $\Psi_\alpha(x_0) = 1$

long term yield $Y(t,\theta) = abx_0$

Normal Yield curve $Y(t,\theta)$ is increasing if $r_t < ab/\Psi'_\alpha(x_0)$

Inverse Yield curve $Y(t,\theta)$ is decreasing if $r_t > b$

Humped Yield curve $Y(t,\theta)$ has one maximum and no minimum if $ab/\Psi'_\alpha(x_0) < r_t < b$

Related results for forward curve.

Jump behavior

- The jumps, especially the large jumps capture the significant changes in the interest rate and may imply the downgrade risk of credit quality.
- Fix $y > 0$ and define the first time that the jump of r is large than $\sigma_Z y$, i.e. $\tau_y = \inf\{t > 0 : \Delta r_t > \sigma_Z y\}$.
- Consider the truncated process $r^{(y)}$ as

$$r_t^{(y)} = r_0 + \int_0^t \widetilde{a}(\alpha, y)\big(\widetilde{b}(\alpha, y) - r_s\big)ds + \sigma \int_0^t \int_0^{r_s} W(ds, du)$$
$$+ \sigma_Z \int_0^t \int_0^{r_{s-}} \int_0^y \zeta \widetilde{N}(ds, du, d\zeta).$$

- It is also a CBI process which coincides with r up to τ_y, and with the branching mechanism given by

$$\Psi^{(y)} = \Psi + \sigma_Z^\alpha \int_y^\infty (1 - e^{-q\zeta})\mu(d\zeta).$$

Probability law of the first large jump

We have

$$\mathbb{P}(\tau_y > t) = \exp\left(-l(y,t)r_0 - ab\int_0^t l(y,s)ds\right)$$

where $l(y,t)$ is the unique solution of

$$\frac{dl}{dt}(y,t) = \sigma_Z^\alpha \int_y^\infty \mu(d\zeta) - \Psi^{(y)}(l(y,t)),$$

with initial condition $l(y,0) = 0$.

- Equivalent form :

$$\mathbb{P}(\tau_y > t) = \mathbb{E}\left[\exp\left\{-\sigma_Z^\alpha\left(\int_y^\infty \mu(d\zeta)\right)\left(\int_0^t r_s^{(y)} ds\right)\right\}\right].$$

which is a bond price written on the auxiliary rate $r^{(y)}$ weighted by the measure μ restricted on (y,∞).

Thanks for your attention !

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