Consistent Utility of Investment and Consumption : a forward/backward SPDE viewpoint.

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OUTLINE

INTRODUCTION

An economic motivation

- the consumption rate : a key process in the economic modeling,
- among these economic literature involving the optimization of the utility of the consumption, many papers focus on long term issues.
  - Economists agree on the necessity of a sequential decision scheme that allows to revise the first decisions and preferences in the light of new knowledge and direct experiences.
  - The utility criterion must be adaptative and adjusted to the information flow.

Consistent progressive utility of investment

- Musiela and Zariphopoulou (2007,2010) were the first to suggest to use instead of the classic criterion the concept of progressive dynamic utility
- The dynamic utility must be consistent with respect to a given investment universe.
- El Karoui and Mrad (2013,2014): existence and characterization of consistent dynamic utility studied from a PDE point of view (in the general setting)

Consistent progressive utility of investment and consumption

- Berrier and Tehranchi (2011) : first order consistency conditions and explicit characterization of consistent stochastic utilities of investment and consumption in the case without volatility vector for the utility process.

Our aims:

- extend this characterization in a general semimartingale setting for the utility process,
- show how the utility of investment and the utility of consumption must be linked in order to ensure the consistency.
- study the similarities and the differences between progressive utilities and the value function of a backward standard utility optimization problem

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Investment set with consumption

Incomplete Itô market, defined on a filtered probability space $(\Omega, (\mathcal{F}_t), \mathbb{P})$ driven by a n-standard Brownian motion W.

Market Parameters

- d risky assets, $d \leq n$.
- $(r_t)_{t\geq 0}$, $(\eta_t)_{t\geq 0}$, $(\sigma_t)_{t\geq 0}$ adapted processes.
- $r_t \geq 0$ spot rate.
- η_t n-dimensional risk premium vector.
- σ_t volatility process $d \times n$.

Utility function

- u strictly concave, strictly increasing, non-negative function on $\mathbb{R}^+$,
- Continuous *marginal utility* u_x, satisfying the Inada conditions $\lim_{x \mapsto \infty} u_x(x) = 0$ and $\lim_{x \mapsto 0} u_x(x) = \infty$.
- Relative risk aversion coefficient $-u_{xx}(x)/(xu_x(x))$.
- Asymptotic elasticity $AE(u) = \limsup_{x \mapsto \infty} xu_x(x)/u(x)$
- Convention: small letters for deterministic utilities, capital letters for stochastic utilities.

Admissible consumption plan

- Admissible strategy (κ, ρ).
  - $c_t = \rho_t X_t \geq 0$: **consumption rate**.
  - $\kappa_t := \sigma_t . \pi_t$ with π_t : fractions of the wealth X_t invested in the risky assets.
- Constraints on the portfolio $\Rightarrow$ Incompleteness of the market.
  $\kappa_t \in \mathcal{R}_t$ where $\mathcal{R}_t$ adapted subvector spaces in $\mathbb{R}^n$.
  Typically $\mathcal{R}_t = \sigma_t(\mathbb{R}^d)$, $d \leq n$.
- Self financing dynamics of wealth process with risky portfolio κ and relative consumption rate ρ is given by

$$dX_t^{\kappa,\rho} = X_t^{\kappa,\rho}[(r_t - \rho_t)dt + \kappa_t(dW_t + \eta_t dt)], \quad \kappa_t \in \mathcal{R}_t. \tag{1}$$

  Remark : $\kappa_t.\eta_t = \kappa_t.\eta_t^{\mathcal{R}}$ where $\eta^{\mathcal{R}}$ is the "minimal" risk premium.
- $\mathscr{X}^c$:= set of wealth processes with admissible (κ_t, ρ_t), and $\mathscr{X}^c(\tau, \xi)$ if the portfolios are starting at a stopping time τ from the initial wealth $\xi \in \mathcal{F}_\tau$

State price density process

- Y^ν is called an admissible state price density process if for any admissible (wealth, consumption)-processes $(X^{\kappa,\rho}, c)$ with $c = \rho X^{\kappa,\rho}$,

  the process $\mathcal{H}_t^{\kappa,\rho,\nu} = X_t^{\kappa,\rho}\, Y_t^\nu + \int_0^t Y_s^\nu c_s\, ds$ is a local martingale.

- It follows that the differential decomposition of Y^ν does not depend on c and

$$dY_t^\nu = Y_t^\nu[-r_t dt + (\nu_t - \eta_t^{\mathcal{R}}).dW_t], \quad \nu_t \in \mathcal{R}_t^\perp, \quad Y_0^\nu = Y_0. \tag{2}$$

- $\mathscr{Y}$:= the convex family of all state density processes Y^ν where $\nu \in \mathcal{R}^\perp$.
  $\mathscr{Y}(\tau, \psi)$: the subfamily of the processes starting from $\psi \in \mathcal{F}_\tau$ at time τ.

STANDARD CONSUMPTION-PORTFOLIO OPTIMIZATION PROBLEM

Standard problem of optimizing expected utility of consumption and terminal wealth with

- a given horizon T_H,
- two deterministic utility functions $u(.)$ and $v(t,.)$

Value function

$$\mathcal{U}(x) = \sup_{(\kappa,\rho)\in\mathscr{X}^c(x)} \mathbb{E}\Big(u(X_{T_H}^{\kappa,\rho}) + \int_0^{T_H} v(t,c_t)dt\Big). \tag{3}$$

- It relies in the use of duality relationships in the spaces of convex functions and semimartingales, together with analysis tools.
- It requires the assumptions that the **asymptotic elasticity** of u is strictly less than one $AE(u) = \limsup_{x\mapsto\infty} xu_x(x)/u(x) < 1$.
- Problem without consumption: Kramkov and Schachermayer (1999, 2003)
- Problem with consumption: Karatzas and Shreve (2001)
- Random terminal utilities: Karatzas and Žitković (2003)

Dynamic programming principle

Dynamic programming principle (El Karoui 1979)
For any pair $\tau \leq \vartheta \leq T_H$ of stopping times,

$$\mathcal{U}(\tau,\xi) = \operatorname*{ess\,sup}_{(\kappa,\rho)\in\mathscr{X}^c(\tau,\xi)} \mathbb{E}\Big(\mathcal{U}(\vartheta, X^{\kappa,\rho}_{\vartheta}(\tau,\xi)) + \int_\tau^\vartheta v(s,c_s)ds|\mathcal{F}_\tau\Big)\ a.s. \qquad (4)$$

- For any admissible (κ,ρ) and any initial condition (τ,ξ), on $[\tau, T_H]$, the preference process $\mathcal{Z}^{\kappa,\rho}$ is a supermartingale, where

$$\mathcal{Z}^{\kappa,\rho}_t(\tau,\xi) = \mathcal{U}(t, X^{\kappa,\rho}_t(\tau,\xi)) + \int_\tau^t v(s,c_s)ds. \qquad (5)$$

- There exists an optimal strategy $(\kappa^{*,H}_{.}(\tau,\xi), c^{*,H}_{.}(\tau,\xi), X^{*,H}_{.}(\tau,\xi))$, such that the optimal preference process $\mathcal{Z}^{*,H}_t(\tau,\xi)$ is a **martingale**

$$\mathcal{Z}^{*,H}_t(\tau,\xi) = \mathcal{U}(t, X^{*,H}_t(\tau,\xi)) + \int_\tau^t v(t, c^{*,H}_t(\tau,\xi))ds. \qquad (6)$$

THE DUAL PROBLEM

The standard state price density conjugate problem

$$\widetilde{\mathcal{U}}(\tau,\psi) = \operatorname{ess\,inf}_{Y^\nu \in \mathscr{Y}(\tau,\psi)} \mathbb{E}\Big(\tilde{u}(T_H, Y^\nu_{T_H}) + \int_\tau^{T_H} \tilde{v}(s, Y^\nu_s) ds | \mathcal{F}_\tau \Big), \ a.s. \quad (7)$$

- Fenchel-Legendre convex conjugate $\tilde{u}(y) = \sup_{x>0} \left(u(x) - yx \right)$.
- From the maximum principle, with $\mathcal{U}_x(\tau,\xi) = \psi$

$$\mathcal{U}_x(t, X^{*,H}_t(\tau,\xi)) = v_c(t, c^{*,H}_t(\tau,\xi)) = Y^{*,H}_t(\tau,\psi). \quad (8)$$

Forward/Backward viewpoints

Proximity between the backward and the forward viewpoint, although differences exist in the interpretation and in the mathematical treatment.

In the backward setting

- comparison arguments to justify martingale properties
- Obtaining closed formula and explicit construction for these value functions and their optimal strategies is a difficult task, except for a few cases like exponential or power utilities.
- Optimal processes are highly dependent on the horizon T_H, which leads to inter-temporal issues.

In the forward setting

- The progressive approach relies on a calibration viewpoint, given a set of test processes.
- The problem is posed forward, leading to time-coherent optimal/extremal processes.
- puts emphasis on the monotonicity of optimal processes with respect to their initial values.

Outline

Learning sets

As in statistical learning, the utility criteria are dynamically adjusted given the family of test processes $\mathscr{X}$, also called the learning set.

Two different admissible learning sets

- $\mathscr{X}^\rho$ "first test problem with fixed ρ": the test processes are associated with any admissible portfolio strategy and a **given** relative consumption rate process ρ_t.
- $\mathscr{X}^c$ "second test problem with consumption": the relative consumption rate ρ is no more given but also a control parameter.

Progressive utilities $(\mathbf{U}, \mathbf{V})$: family of cadlag adapted processes such that $\mathbb{P}$ as, for every $t \geq 0$, $x \to U(t, x)$ and $c \to V(t, c)$ are standard utility functions.

- Since $\mathscr{X}$ is a learning set, there is no satisfaction to invest in the set $\mathscr{X}$, (ie in mean the future is less preferable than the present): *supermartingale property* of the dynamic preference process $(Z_t^{\kappa,\rho})$
- The stochastic utilities $(U(t, x), V(t, c))$ is the best choice: the previous supermartingale Z^e is a martingale for some extremal process (κ^e, ρ^e).

Definition of a consistent progressive utility system

Let $(\mathbf{U}, \mathbf{V})$ be a progressive utility system with learning set $\mathscr{X}$.

- The utility system $(\mathbf{U}, \mathbf{V})$ is said to be $\mathscr{X}$-consistent, if for any admissible test process $X^{\kappa,\rho} \in \mathscr{X}$, the preference process

$$Z_t^{\kappa,\rho} = U(t, X_t^{\kappa,\rho}) + \int_0^t V(s, c_s)ds \text{ is a non-negative } \textbf{supermartingale}. \quad (9)$$

- The consistent utility system $(\mathbf{U}, \mathbf{V})$ is said to be $\mathscr{X}$-strongly consistent if there exists an extremal system in $\mathscr{X}$, (κ^e, ρ^e), with $c^e = \rho^e X^e$ and $X^e = X^{\kappa^e, \rho^e}$, binding the constraint, that is the extremal preference process

$$Z_t^e = U(t, X_t^e) + \int_0^t V(s, c_s^e)ds \quad \text{is a } \textbf{martingale}. \quad (10)$$

- Example: The value function system $(\mathcal{U}(t,x), v(t,c))$ of the classic consumption optimization problem is a strongly consistent system (with respect to $\mathscr{X}^c$), **defined from its terminal condition** $\mathcal{U}(T_H, x) = u(x)$

Differential point of view for Itô consistent utility system

In the forward case, in the absence of Markov property, stochastic calculus can be used to characterize $\mathscr{X}$-consistent forward utility system, via a stochastic generalization of the deterministic backward HJB-PDE.

- General assumption that the utility random field **U** is a "regular" Itô random field with differential decomposition

$$dU(t,x) = \beta(t,x)dt + \gamma(t,x).dW_t \tag{11}$$

- $\beta(t,x)$ is the **drift random field**
- $\gamma(t,x)$ is the **multivariate diffusion random field**.

• How **to read directly on its local characteristics** (β, γ) **that the process** $U(t,x)$ **is a utility random field (increasing and concave)**?

- Study of the forward marginal utility $\mathbf{U_x}$
- The solution of the SDE(μ, σ) with "regular" coefficients

$$dX_t = \mu(t, X_t)dt + \sigma(t, X_t).dW_t \tag{12}$$

is monotonic w.r.t. its initial condition.

• How to **express the supermartingale property implied by the consistency condition in terms of local characteristics** ?

- differential decomposition of $U(t, X_t^{\kappa,\rho})$, where **U** is now a dynamic random field
- Itô-Ventzel's formula: Let $(G(t,x))$ "regular" random field with local characteristics $\phi(t,x)$ and $\psi(t,x)$. Then

$$\begin{aligned} dG(t,X_t) &= \left(\phi(t,X_t)\,dt + \psi(t,X_t).dW_t\right) \\ &+ \left(G_x(t,X_t)dX_t + \frac{1}{2}G_{xx}(t,X_t)d < X, X >_t\right) \\ &+ \left(< \psi_x(t,X_t)dW_t, dX_t >\right). \end{aligned}$$

SPDE point of view

Let $(X(t,x))$ **be the monotonic solution of a "regular" SDE**(μ,σ), with adjoint operator in divergence form,

$$\widehat{L}^{\mu,\sigma}_{t,z} = \frac{1}{2}\partial_z(\|\sigma(t,z)\|^2\partial_z) - \mu(t,z)\partial_z.$$

- The inverse flow $\xi(t,z) = X^{-1}(t,z)$ is "regular" and solution of the SPDE

$$d\xi(t,z) = -\xi_z(t,z)\sigma(t,z).dW_t + \widehat{L}^{\mu,\sigma}_{t,z}(\xi)dt, \quad \xi(0,z) = z. \tag{13}$$

- Let Y be a "regular" solution of the SDE(μ^Y,σ^Y) and ϕ any $\mathcal{C}^2$-function. Then the **compound random field** $H(t,z) := Y(t,\phi(\xi(t,z)))$ with initial condition $H(0,z) = \phi(z)$ evolves as

$$\begin{aligned} dH(t,z) &= \big(\sigma^Y(t,H(t,z)) - H_z(t,z)\sigma^X(t,z)\big).dW_t \qquad (14)\\ &+\big(\mu^Y(t,H(t,z)) - H_z(t,z)\sigma^X(t,z).\sigma^Y_y(t,H(t,z)) + \widehat{L}^{\sigma,\mu}_{t,z}(H)(t,z)\big)dt. \end{aligned}$$

Two test problems

Let $(\mathbf{U},\mathbf{V})$ be a "regular utility" system and (β,γ) the local characteristics of $\mathbf{U}$. The **extremal diffusion coefficient** is defined by

$$\sigma^e(t,x) = x\kappa^e_t(x) = -\frac{U_x(t,x)}{U_{xx}(t,x)}\big(\eta^{\mathcal{R}}_t + \frac{\gamma^{\mathcal{R}}_x(t,x)}{U_x(t,x)}\big). \tag{15}$$

- First test problem with fixed ρ. The utility system $(\mathbf{U},\mathbf{V})$ is consistent with the family of test processes $\mathscr{X}^\rho = \{X^{\kappa,\rho}, | \kappa \in \mathcal{R}, \rho \text{ given}\}$ if

$$\beta(t,x) = -U_x(t,x)x(r_t - \rho_t) + \frac{1}{2}U_{xx}(t,x)\|\sigma^e(t,x)\|^2 - V(t,\rho_t x). \tag{16}$$

- Second test problem with consumption. $(\mathbf{U},\mathbf{V})$ is consistent with the family of test processes $\mathscr{X}^c = \{X^{\kappa,\rho}, | \kappa \in \mathcal{R}, \text{any process } \rho > 0\}$ if

$$\beta(t,x) = -U_x(t,x)xr_t + \frac{1}{2}U_{xx}(t,x)\|\sigma^e(t,x)\|^2 - \tilde{V}(t,U_x(t,x)) \tag{17}$$

where $\tilde{V}(t,z) = \sup_{\rho>0}(V(t,\rho) - z\rho)$ is the Fenchel transform of V. The **extremal consumption** $(\rho^e(t,x)x)$ is given by

$$\rho^e(t,x)x = V_z^{-1}(t,U_x(t,x)) = -\tilde{V}_z(t,U_x(t,x)).$$

Marginal utility

Let $(\mathbf{U}, \mathbf{V})$ be a "regular utility" system, with local characteristic (β, γ) for $\mathbf{U}$.

- The HJB-constraint

$$\beta(t,x) = -U_x(t,x)xr_t + \frac{1}{2}U_{xx}(t,x)\|x\kappa_t^e(x)\|^2 - \tilde{V}(t, U_x(t,x)) \tag{18}$$

is equivalent to the following property of the marginal utility $U_x(t,x)$ with characteristics (β_x, γ_x)

$$\begin{cases} \beta_x(t,x) = \mu^Y(t, U_x(t,x)) - U_{xx}(t,x)\sigma^e(t,x).\sigma_y^Y(t, U_x(t,x)) + \widehat{L}_{t,x}^{(\mu^e,\sigma^e)}(U_x) \\ \gamma_x(t,x) = -U_{xx}(t,x)\sigma_t^e(x) + \sigma^Y(t, U_x(t,x)) \end{cases}$$

with

$$\begin{cases} \sigma^e(t,x) &= -\frac{U_x(t,x)}{U_{xx}(t,x)}\left(\eta_t^{\mathcal{R}} + \frac{\gamma_x^{\mathcal{R}}(t,x)}{U_x(t,x)}\right),\ \mu_t^e(x) = r_t x + \sigma_t^e(x).\eta_t^{\mathcal{R}} - x\rho^e(t,x). \\ \sigma^Y(t,y) &= \gamma_x^{\perp}(t, U_x^{-1}(t,y)) - y\eta^{\mathcal{R}}, \quad \mu^Y(t,y) = -r_t y. \end{cases}$$

- Then the marginal utility $U_x(t,x)$ has the characteristics of a compound random field generated by the two SDEs, SDE(μ^e, σ^e) and SDE(μ^Y, σ^Y)

$$U_x(t,x) = Y^e(t, u_x((X^e)^{-1}))(t,x), \quad V_c(t,c) = U_x(t, (x\rho^e(t,x))^{-1})(t,c). \tag{19}$$

CONSUMPTION UTILITIES COMPATIBLE WITH COHERENT POWER UTILITIES

- A consumption consistent progressive power utility system is necessarily a pair of power utilities with the same risk aversion coefficient $\alpha < 1$:

$$U^{(\alpha)}(t,x) = Z_t \frac{x^{1-\alpha}}{1-\alpha} = Z_t u^{(\alpha)}(x) \quad \text{and} \quad V^{(\alpha)}(t,c) = (\frac{\bar{v}_t}{\alpha})^{\alpha} U^{(\alpha)}(t,x)$$

whose coefficient Z_t satisfies the HJB drift constraint,

$$dZ_t = -Z_t\big[((1-\alpha)r_t + \frac{1-\alpha}{2\alpha}\|\eta_t^{\mathcal{R}} + \delta_t^{Z,\mathcal{R}}\|^2 + \bar{v}_t)dt - \delta_t^Z.dW_t\big]. \qquad (20)$$

- The extremal processes are linear with respect to their initial conditions

$$X_t^e(x) = xX_t^e,\ Y_t^e(y) = yY_t^e,\ \sigma_t^e(x) = x\kappa_t^e = x\frac{\|\eta_t^{\mathcal{R}} + \delta_t^{Z,\mathcal{R}}\|^2}{\alpha},\ c_t^e(z) = z\rho_t^e = z\frac{\bar{v}_t}{\alpha}$$

$$dX_t^e = X_t^e\big((r_t - \rho_t^e)dt + \kappa_t^e.(dW_t + \eta_t^{\mathcal{R}}dt)\big), dY_t^e = Y_t^e\big(-r_t dt + (\nu_t^e - \eta_t^{\mathcal{R}}).dW_t\big).$$

Moreover $Z_t = Y_t^e(X_t^e)^{\alpha} = Y_t^e u_x^{(\alpha)}(X_t^e)$.

POWER UTILITIES: THE BACKWARD CASE

Backward consumption-portfolio optimization problem with

- time horizon T_H,
- terminal wealth utility $\zeta_{T_H}\, u^{(\alpha)}(x)$,
- a consumption utility $v(t,c) = \phi_t u^{(\alpha)}(x)$, where ϕ_t is a given process.

Assume the value function $\mathcal{U}(0,x)$ well-defined at time 0.

- The value system $(\mathcal{U}(t,x), v(t,c))$ is a consistent power utility system $(Z_t u^{(\alpha)}(x), \phi_t u^{(\alpha)}(x))$, if there exists a solution (Z_t, δ_t^Z) of the backward SDE

$$\begin{aligned} dZ_t &= -Z_t\big[((1-\alpha)r_t + \frac{1-\alpha}{2\alpha}\|\eta_t^{\mathcal{R}} + \delta_t^{Z,\mathcal{R}}\|^2 + \alpha(\frac{\phi_t}{Z_t})^{1/\alpha})dt - \delta_t^Z.dW_t\big], \\ Z_{T_H} &= \zeta_{T_H}. \end{aligned}$$

- Then all the properties given in the forward case hold true.

OUTLINE

CONCLUSION

We provide an extension of the notion of consistent progressive utilities **U** to consistent progressive utilities of investment and consumption (**U**, **V**)

- An example: the value function of the classic **backward** setting with a given (deterministic) terminal utility.
- Market consistency in the **forward** setting:
  U satisfies a **SPDE of Hamilton-Jacobi-Bellman (HJB)-type**.
- This SPDE highlights the links
  - between the utility of wealth **U** and the utility of consumption **V**,
  - between the drift and the volatility characteristics of the utility **U**.
- By associating with the HJB-SPDE two SDEs, we discuss the existence and the uniqueness of a concave solution.
- We provide explicit regularity conditions and characterize the consistent pairs of consistent utilities of investment and consumption.
- Examples of power utilities.

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