Multi-dimensional BSDEs whose terminal values are bounded and have bounded Malliavin derivatives

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We talk about the BSDE calculus in the very classical sense. We would like to have this talk, because there are some points in the BSDE calculus, which seem fundamental and elementary, but which seem absent from the current literature. It is a good occasion to check them before an aware public.

Our work has begun from the following observation. In recent years, there has been a lot of progress on the multi-dimensional non-Lipschitzian BSDEs. At the same time, because of the high technicality of the computations, this development becomes hard to follow for a non specialist. The difficulty is notably caused by the lack of a global and centralized vision of all the techniques involved. The idea arises then to make a representation of all the techniques in form of a network organized according to the way the techniques are involved in computation, according to the reasons for which they are introduced, according to the consequences they yield, and so on. It is expected that such an effort helps to better understand the multi-dimensional non Lipschitzian BSDEs.

Many aspects can be used to classify the notions and the computing techniques. We can mention, for example, the Markov property, the Girsanov transformation technique, the smallness techniques, the exponential transformation, the Malliavin derivative, the Lyapunov function, the sliceability, etc.. In the present talk, we focus on the class of the multi-dimensional BSDEs whose terminal values are bounded as well as their corresponding Malliavin derivatives.

Before continuing with the network of techniques, let us present the precise conditions which define the classes of BSDEs we consider in this talk. Here is the first package of conditions (about a d-dimensional BSDE$[T,\xi,f]$ with respect to a n-dimensional Brownian motion):

$$
\begin{array}{ll}
- & \text{the driver } f(t,y,z) \text{ has first derivatives in } y,z \text{ and the derivatives} \\
  & \text{are continuous in } y,z, \\
- & \text{for every } i,t,y,z,\ f(i,t,y,z) \text{ is Malliavin differentiable in } \mathbb{D}^{1,2}, \\
- & \text{the function } D_{j,\theta}f(\omega,i,t,y,z) \text{ is measurable}, \\
- & \text{for every } j,\theta,i,y,z, \text{ the process } D_{j,\theta}f(i,\cdot,y,z) \text{ is predictable}, \\
- & \text{for every } y,z,\ \mathbb{E}[\int_0^T\int_0^T \|D_\theta f(s,y,z)\|^2 ds\ d\theta] < \infty, \\
- & \text{for every } j,\theta,i,t, \text{ the random function } D_{j,\theta}f(i,t,y,z) \text{ is Lipschitzian in } y,z \\
  & \text{with common index } K(\theta,t) \text{ such that } \int_0^T \mathbb{E}[(\int_0^T K(\theta,s)^2 ds)^2] d\theta < \infty.
\end{array}
\tag{1}
$$

We recall that, under the conditions (1) and

$$
\begin{array}{ll}
- & \xi \in \mathbf{L}^4, \\
- & \text{the derivatives of } f(i,t,y,z) \text{ in } y,z \text{ are bounded}, \\
- & \sum_{i=1}^{d} \mathbb{E}[(\int_0^T f(i,s,0,0)^2 ds)^2] < \infty, \\
- & \mathbb{E}[\int_0^T \int_0^T \|D_\theta f(s,Y_s,Z_s)\|^2 ds \ d\theta] < \infty,
\end{array}
\tag{2}
$$

where (Y,Z) is the solution of the BSDE$[T,\xi,f]$, which exists, because the boundedness of the derivatives of f implies that the driver f is Lipschitzian, ElKarouis-Peng-Quenez (1997) has proved the Malliavin differentiability of the BSDE$[T,\xi,f]$.

Here also, the conditions (1) will be used to ensure the Malliavin differentiability of the BSDEs. Notice that we will not discuss if the conditions (1) are optimal. We simply recall that other studies exist on the Malliavin differentiability of BSDEs, and, if we replace the paper ElKarouis-Peng-Quenez (1997) by other conditions of Malliavin differentiability, our results remain valid.

Let $\varkappa$ be a (deterministic) function defined on $\mathbb{R}_+$ which is continuous, nonnegative, increasing and locally Lipschitzian. We say that f has an increment rate in z uniformly less than $\varkappa$, if, for $1 \leq i \leq d$, for $y \in \mathbb{R}^d$, for $z, z' \in \mathbb{R}^{d \times n}$,

$$|f(i,t,y,z) - f(i,t,y,z')| \leq \varkappa(\|z\| + \|z'\|)\|z - z'\|. \tag{3}$$

We will call the function $\varkappa$ a rate function in z.

The use of the rate function $\varkappa$ give us the facility to consider simultaneously BSDEs with different increment conditions. When $\varkappa$ is a positive constant $\eta > 0$, we have the uniform Lipschitzian condition in z. When $\varkappa$ is an affine function with positive coefficients, we have a quadratic increment condition in z.

We introduce two other packages of conditions:

- $\xi \in \mathcal{F}_T$, $\|\xi\|_\infty < \infty$, $\xi \in \mathbb{D}^{1,2}$ and $\sup_{1\le j\le n} \sup_{1\le i\le d} \sup_{0\le\theta<\infty} \|D_{j,\theta}\xi_i\|_\infty < \infty$,
- the driver $f(i,t,y,z)$ is uniformly Lipschitzian in y with index β and has an increment rate in z less than $\varkappa$, while its root $f(i,t,0,0)$ is bounded by a constant υ, (4)
- the conditions (1) (the differentiability of the driver f),
- $\|D_{j,\theta}f(i,t,y,z)\|_\infty$ is uniformly bounded on any bounded set of (j,θ,i,t,y,z).

And

The Malliavin derivatives $D_{j,\theta}f(i,t,y,z)$, $1\le i\le d, 1\le j\le n, 0\le\theta\le T$, are uniformly Lipschitzian in y with a common index $\widehat{\beta}$ and have increment rates in z less than a common rate function $\widehat{\varkappa}$, while their roots $D_{j,\theta}f(i,t,0,0)$ are bounded by a common constant $\widehat{\upsilon}$. (5)

Clearly there are some redundancy among the above packages of conditions. They are maintained nevertheless separate, because they play different roles in different results.

Notice that the above conditions are almost the minimum conditions that we have to assume to have the results of this talk. For example, the conditions (1) is required to have the Malliavin differentiability of the BSDE. The boundedness of $f(i,t,0,0)$ is required, because we hope that the solution of the BSDE preserves the boundedness of ξ.

Let us turn back to the idea to represent the computing techniques in form of a network. In drawing the network, one perceive centralized configurations around some hidden nodes. Highlighting these nodes would rend the network better understandable.

It is the case with the following point. Like the ordinary differential equations, the multi-dimensional non Lipschitzian BSDEs often have solutions locally, but the solutions can explode at finite horizons. Hence, the study of a multi-dimensional BSDEs should not be viewed only as the search of solutions for a given horizon. Rather, the essential part of this study consists in analyse local solutions and computing the corresponding explosion times. The two point of view are different in that one tries to control the whole by *a priori inequalities*, while the other one regards the infinitesimal action of the BSDE on the random variables (a g-expectation point of view).

In this regard, we prove the following result. We need some vocabulary to state the theorem.

Consider the BSDE$[T,\xi,f]$. If a solution (Y,Z) of the BSDE$[T,\xi,f]$ exists on $[t,T]$ as well as its corresponding Malliavin derivatives, we define, for $0 \leq t \leq T$,

$$\begin{aligned} \Lambda_t &= \sup_{t\leq s\leq T} \textstyle\sum_{i=1}^{d} \|Y_{i,s}\|_\infty, \\ \widehat{\Lambda}_t &= \sup_{t\leq s\leq T} \sup_{0\leq\theta<\infty} \|\sqrt{\textstyle\sum_{j=1}^{n}\sum_{i=1}^{d}(D_{j,\theta}Y_{i,s})^2}\|_\infty. \end{aligned} \tag{6}$$

We consider the BSDEs which satisfy the conditions (4) and (5). Let $\varkappa$ and $\widehat{\varkappa}$ be the two rate functions given in (4) and (5). Define $\mathrm{g}^\circ(t,u), 0 \leq t \leq T, 0 \leq u < \infty$, to be the real function:

$$\begin{aligned} & u \times \left(d \times 2\beta + d^2 \times n \times \varkappa(2\sqrt{u})^2\right) \\ & +\sqrt{u}2\sqrt{d\times n}\big(\widehat{\upsilon} + \widehat{\beta}e^{d\times\beta\times(T-t)}\big(\textstyle\sum_{i=1}^{d}\|\xi_i\|_\infty + d\times(\upsilon + \varkappa(\sqrt{u})\sqrt{u})(T-t)\big) + \widehat{\varkappa}(\sqrt{u})\sqrt{u}\big). \end{aligned}$$

Let $\mathrm{g}(t,u)$ be any continuously differentiable function, defined on $(t,u) \in [0,T] \times \mathbb{R}_+$, such that $\mathrm{g}(t,u) \geq \mathrm{g}^\circ(t,u)$. Consider the (ordinary) differential equation, defined on $[0,T]$:

$$\begin{cases} u_0 = \sup_{0 \leq \theta < \infty} \|D_\theta \xi\|_\infty^2, \\ \frac{du_t}{dt} = \mathrm{g}(T-t, u_t). \end{cases} \tag{7}$$

Let α_{g} be the supremum of the $0 \leq a \leq T$ such that the differential equation (7) has a finite classical solution on $[0,a]$. Let u_{g} be the solution of the differential equation (7) on the time interval $[0,\alpha_{\mathrm{g}})$.

A solution (Y, Z) of the BSDE$[T, \xi, f]$ is said to satisfy the $\mathfrak{M}d$-property, if (Y, Z) is in the space $\mathcal{S}_\infty[0, T] \times \mathcal{Z}_{\text{BMO}}[0, T]$ and

- the Malliavin derivatives $D_{j,\theta} Y_t, 1 \leq j \leq n, 0 \leq \theta \leq T, 0 \leq t \leq T$, exist and
- $D_\theta Y_\theta = Z_\theta$ for $0 \leq \theta \leq T$, and
- the processes $(D_{j,\theta} Y)_{1 \leq j \leq n, 0 \leq \theta \leq T}$ form a bounded family in the space $\mathcal{S}_\infty[0, T]$.

Clearly, this notion can be defined on any sub-interval of $[0, T]$.

We now state the theorem.

Theorem 1 *Suppose that the parameters of the BSDE*$[T, \xi, f]$ *satisfy the conditions (4) and (5). Then, for any* $T - \alpha_{\mathrm{g}} < a \leq T$, *a unique solution* (Y, Z) *of the BSDE*$[T, \xi, f]$ *exists in* $\mathcal{S}_\infty[a, T] \times \mathcal{Z}_{BMO}[a, T]$ *on the time interval* $[a, T]$ *with* $\mathfrak{M}d$*-property. On the time interval* $(T - \alpha_{\mathrm{g}}, T]$, *the functions* Λ *and* $\widehat{\Lambda}$ *are well-defined and continuous and satisfy*

$$\widehat{\Lambda}_t^2 \leq u_{\mathrm{g}, T-t} \quad \text{and}$$
$$\Lambda_t \leq e^{d \times \beta \times (T-t)} \big(\textstyle\sum_{i=1}^d \|\xi_i\|_\infty + d \times (\upsilon + \varkappa(\widehat{\Lambda}_t)\widehat{\Lambda}_t)(T-t) \big).$$

Actually, we obtain more: $\limsup_{t' \uparrow t} \frac{1}{t-t'} \big(\widehat{\Lambda}_{t'}^2 - \widehat{\Lambda}_t^2 \big) \leq \mathrm{g}(t, \widehat{\Lambda}_t^2)$ on the time interval $(T - \alpha_{\mathrm{g}}, T]$, and the proof of the theorem is based on this differential inequality.

From the very beginning, one knows that the main problem in the BSDE calculus is the control of the integrability of the solution processes (the *a priori inequalities*). Various integrability conditions are introduced in the BSDEs calculus. In particular, the exponential integrability has been very useful in the computation of the one-dimensional quadratic BSDEs. However, the exponential integrability is largely absent from the computations of multi-dimensional quadratic BSDEs (partially because the exponential transformations become inefficient). This is the second hidden node that we want speak about: the exponential integrability plays always a central role, even in the multi-dimensional BSDE calculus.

We have the following result. We introduce some vocabulary. For any positive real number $v \in \mathbb{R}^*$, let $r_v(x), x \in \mathbb{R}$, denotes a non decreasing odd C_b^2-function such that $|r_v'(x)| \leq 1$ and

$$\begin{array}{lll} r_v(x) & = x & \text{for } -v \leq x \leq v, \\ r_v(x) & = -v-1 & \text{for } x \leq -v-4, \\ r_v(x) & = v+1 & \text{for } x \geq v+4. \end{array}$$

For a matrix $z = (z_{i,j}) \in \mathbb{R}^{d \times n}$, we define $\mathbf{r}_v(z)$ to be the matrix of $r_v(z_{i,j})$. We can verify the inequalities:

$$\|\mathbf{r}_v(z)\| \leq \|z\|, \quad \|\mathbf{r}_v(z') - \mathbf{r}_v(z)\| \leq \|z' - z\|.$$

We introduce the modified drivers

$$\bar{f}_N(t,y,z) = f(t,y,\mathbf{r}_N(z)), \quad N \in \mathbb{N}^*, y \in \mathbb{R}^d, z \in \mathbb{R}^{d\times n}.$$

Clearly, f is the pointwise limit of $\bar{f}_N$. If $f(t,y,z)$ is Malliavin differentiable, we have $D\bar{f}_N(t,y,z) = Df(t,y,\mathbf{r}_N(z))$. It is to note also that, if f is a driver with Lipschitzian condition in y and $\varkappa$-rate condition in z, for all $N > 0$, $\bar{f}_N$ is a Lipschitzian driver in the two variables y, z. Especially that, if the BSDE$[T,\xi,\bar{f}_N]$ with the driver $\bar{f}_N$ has a solution (Y_N, Z_N) such that $\sup_{0\le s\le T} \|Z_{N,s}\|_\infty < N$, the process $(Y,Z) := (Y_N, Z_N)$ is actually a solution of the BSDE$[T,\xi,f]$ (with the original driver f), and vice versa (a very strong stability feature).

For any $N \in \mathbb{N}^*$ let (Y_N, Z_N) be the solution of the BSDE$[T, \xi, \bar{f}_N]$. We introduce the conditions:

$$\begin{array}{l} \sup_{N\in\mathbb{N}^*, 0\le t\le T} \| \ \mathbb{E}[e^{28d^2\times n \int_t^T \varkappa(2\|Z_{N,s}\|)^2 ds}|\mathcal{F}_t] \ \|_\infty < \infty, \\ \sup_{N\in\mathbb{N}^*, 1\le j\le n, 0\le\theta<\infty, 1\le i\le d, 0\le t\le T} \|\mathbb{E}[\ \int_t^T \ |D_{j,\theta}\bar{f}_N(i, s, Y_{N,s}, Z_{N,s})|ds \ |\mathcal{F}_t]\|_\infty < \infty. \end{array} \tag{8}$$

Theorem 2 *Suppose that the parameters of the BSDE*$[T, \xi, f]$ *satisfy the conditions (4). Then, the BSDE*$[T, \xi, f]$ *has a unique solution* (Y, Z) *in* $\mathcal{S}_\infty[0, T] \times \mathcal{Z}_{BMO}[0, T]$ *with the* $\mathfrak{M}d$*-property, if and only if the conditions (8) hold. In this case, the Malliavin derivatives* $D_{j,\theta}Y$ *satisfy a linear BSDE on* $[\theta, T]$.

With the very strong stability of the approximation sequence of BSDE$[T, \xi, \bar{f}_N]$, Theorem 2 is proved with the linear BSDE which governs the Malliavin derivatives of the solution of a BSDE.

It comes the third hidden node in our network of techniques: the resolution of linear BSDEs. In the literature, the linear BSDEs are considered as simple and are presented as annex to the general BSDE calculus, which is much influenced by the Lipschitzian consideration (i.e., the bounded coefficients). It is not a satisfactory situation.

In fact, in our study, we have needed to consider the BSDEs as transformations on random variables (a g-expectation consideration). We need to know what is the optimal condition on the coefficients, under which a linear BSDE transforms a bounded random variable onto a bounded random variable. We have not yet achieved this goal. But we have obtain a sufficient condition much better than the boundedness condition on the coefficients: the exponential integrability, and which is just enough for the proof of Theorem 2.

We introduce some vocabulary. Consider

$$\begin{cases} dY_t = -(f(t,0,0) + g_t Y_t + h_t Z_t)dt + Z_t dB_t, \\ Y_T = \xi, \end{cases} \tag{9}$$

where $g_t = (g_{i,i',t})_{1\leq i,i'\leq d}$ and $h_t = (h_{i,i',j,t})_{1\leq i,i'\leq d, 1\leq j\leq n}$ are matrix of locally bounded predictable processes and

$$g_{i,t}Y_t = \sum_{i'=1}^{d} g_{i,i',t}Y_{i',t}, \;\; h_{i,t}Z_t = \sum_{j'=1}^{n}\sum_{i'=1}^{d} h_{i,i',j',t}Z_{i',j',t}.$$

For constant $p > 0$ let us define the exponential functional

$$\varrho_p(h) := \sup_{0\leq t\leq T} \| \; \mathbb{E}[e^{p\int_t^T \sum_{j=1}^n \|h_{j,s}\|^2 ds}|\mathcal{F}_t] \; \|_\infty. \tag{10}$$

Let $(\phi_{a,t})_{a\leq t\leq T}$ be the solution of

$$d\phi_{a,t} = \phi_{a,t}g_t dt + \textstyle\sum_{j=1}^n \phi_{a,t}h_{j,t}dB_{j,t}, \quad \phi_{i,i'',a,a} = 1\!\!1_{\{i=i''\}}.$$

Lemma 3 *Suppose that $\varrho_{28}(h) < \infty$. Suppose that the terminal value ξ is bounded. Suppose that the components $g_{i,j}$ are bounded by $\beta > 0$ and*

$$\|f\|_{.} = \sup_{1\leq i\leq d} \sup_{0\leq t\leq T} \| \ \mathbb{E}[\ \int_t^T |f(i,s,0,0)|ds\ |\mathcal{F}_t] \ \|_\infty < \infty.$$

Then, for $1 \leq i \leq d$, the semimartingale

$$Y_{i,a} \ = \ \mathbb{E}[\phi_{i,a,T}\xi + \int_a^T \sum_{i''=1}^d \phi_{i,i'',a,s} f(i'',s,0,0)ds|\mathcal{F}_a], \ 0 \leq a \leq T,$$

is a well-defined process and the process Y, together with its martingale coefficient Z, forms a solution of the linear BSDE(9). The process (Y,Z) is the unique solution of the linear BSDE(9) which satisfies $\sup_\sigma \|Y_\sigma\|_{\mathbf{L}^2} < \infty$, where σ runs over all stopping times $\sigma \leq T$. We have the upper bound

$$\| \sum_{i=1}^d |Y_{i,a}| \|_\infty \leq d e^{d\times\beta(T-a)} \varrho_2(h)^{1/4} \frac{4}{3} \varrho_{28}(h)^{\frac{1}{8}} \ (\|\xi\|_\infty + \sqrt{2} d \|f\|_{.}), \ \forall 0 \leq a \leq T.$$

Let us leave our network of techniques and think about the usefulness of Theorem 1 and Theorem 2.

We will explain the situation with a concrete example. It is occasion also to speak about the two notions: the Lyapunov function and the sliceability, which have been particularly important in the non Lipschitzian BSDE calculus.

We say that the BSDE$[T, \xi, f]$ is sub-quadratic in z, if the driver f has a z-increment rate function $\varkappa(x)$ bounded by constant times $1 + |x|^\alpha$ for a $0 < \alpha < 1$ (the sub-quadratic index).

Before continuing, we have a remark on Theorem 1. It is a theorem proved for whole a class of BSDEs. It is therefore not optimal in particular situations. In the case of a sub-quadratic BSDE, we can take the function $C(1+u)^{1+\alpha}$ ($C > 0$ a constant) as the function $\mathrm{g}(t,u)$ in Theorem 1. Resolve the differential equation (7) associated with this function g. We have

$$\left(\frac{1}{(1+\sup_{0\leq\theta<\infty}\|D_\theta\xi\|_\infty^2)^{-\alpha}-\alpha\times C\times t}\right)^{1/\alpha} = u_t.$$

Therefore, the resolution horizon α_{g}, predicted by Theorem 1 with the differential equation (7) is finite:

$$\alpha_{\mathrm{g}} = \frac{1}{(1+\sup_{0\leq\theta<\infty}\|D_\theta\xi\|_\infty^2)^{\alpha}\times\alpha\times C} < \infty.$$

This estimation of the resolution horizon is not "individually" optimal. Below, we will prove that a lot of sub-quadratic BSDEs can have solutions on any horizon $[0,T]$.

We say that a function $\mathtt{h}$ is a (global) Lyapunov function for the driver f, if the function $\mathtt{h}$ is defined on $\mathbb{R}^d$ taking values in $\mathbb{R}_+$, two-times continuously differentiable, such that

$$\frac{1}{2}\sum_{i=1}^d \sum_{i'=1}^d \partial_{y_i}\partial_{y_{i'}}\mathtt{h}(y)\sum_{j=1}^n z_{i,j}z_{i',j} - \sum_{i=1}^d \partial_{y_i}\mathtt{h}(y) f(i,t,y,z) \quad \geq \quad \|z\|^2 - \mathtt{k}(t,y), \qquad (11)$$

for some Borel function $\mathtt{k}(t,y) \geq 0$, for $\forall 0 \leq t \leq T$, $\forall z \in \mathbb{R}^{d\times n}$, $\forall y \in \mathbb{R}^d$. The function $\mathtt{k}$ will be called a lower bound function of the Lyapunov condition (11). In this talk, we always suppose

$$\begin{array}{l} |\mathtt{k}(t,y) - \mathtt{k}(t,0)| \leq \bar{\beta}\sum_{i=1}^d |y_i| \ \text{ for some } \bar{\beta} > 0 \text{ (the growth condition)}, \\ \|\mathtt{k}\|_. = \sup_{1\leq i\leq d}\sup_{0\leq t\leq T} \|\mathbb{E}[\int_t^T |\mathtt{k}(s,0)|ds \ |\mathcal{F}_t]\|_\infty < \infty \text{ (the root condition).} \end{array} \qquad (12)$$

We have called the function $\mathtt{h}$ Lyapunov function, because of the condition (11). This usage is in coherence with the practice in the literature. It is to notice, however, that the conditions designated by the appellation Lyapunov function can be different from one paper to another. Actually, the Lyapunov functions in the literature have different lower bound functions, and the Lyapunov function in Xing-Zitkovic (2017) expresses a local condition.

The second important notion is the sliceability. We say that a predictable real process X is uniformly sliceable on the time interval $[0, T]$, if, for any $c > 0$, there exists $\delta > 0$, for any $0 \leq a < b \leq a + \delta$, the potential bound

$$\sup_{\sigma \leq T} \|\mathbb{E}[\int_\sigma^T 1\!\!1_{\{a<s\leq b\}}|X_s|ds \ |\mathcal{F}_\sigma]\|_\infty \leq c.$$

(The uniform sliceability can be defined in any sub-interval of $[0, T]$ in a similar way.)

The property of sliceability goes back to the study of Schachermayer (1996) on the BMO martingales. (The uniform sliceability in the present talk is stronger than the sliceability of Schachermayer (1996).) But our result below does not depends on the result of Schachermayer (1996).

Theorem 4 *Suppose that the parameters of the BSDE*$[T, \xi, f]$ *satisfy the conditions (4) and (5). Suppose that the driver* f *is sub-quadratic in* z *and the Malliavin derivatives* $D_{j,\theta} f$ *of the driver are quadratic in* z*. Suppose that the driver* f *has a global Lyapunov function. Then, BSDE*$[T, \xi, f]$ *has a unique solution on the whole time horizon* $[0, T]$ *with* $\mathfrak{M}d$*-property.*

We notice that Theorem 2 gives also an extension of ElKarouis-Peng-Quenez (1997) to a family of non Lipschitzian BSDEs.

Let us see quickly how Theorem 2 and Theorem 1 are used to prove Theorem 4.

Let $\mathfrak{B}$ be the set of $0 \leq b \leq T$ such that the BSDE$[T, \xi, f]$ has a unique solution (Y, Z) in the space $\mathcal{S}_\infty[0,T] \times \mathcal{Z}_{\text{BMO}}[0,T]$ on $[b, T]$ with $\mathfrak{M}d$-property. Let $b_0 = \inf \mathfrak{B}$. Theorem 1 shows $b_0 < T$. The conditions of the theorem imply that $\|Z\|^{2\alpha}$ is uniformly sliceable. As consequence, $\widehat{\Lambda}_{b_0+} < \infty$. This means that the solution of the BSDE$[T, \xi, f]$ on $(b_0, T]$ coincides with that of BSDE$[T, \xi, \bar{f}_N]$ for some $N \in \mathbb{N}$. By continuity, we see that necessarily $b_0 \in \mathfrak{B}$.

We say that $b_0 = 0$, which proves the theorem. If it was not the case, because the BSDE$[b_0, Y_{b_0}, f]$ satisfy the conditions (4) and (5), by Theorem 1 (the existence of local solutions), the set $\mathfrak{B}$ would contain a $0 \leq b < b_0$ which is in contradiction with the definition of b_0. ∎

We present the following nice logical sequence:

$$\begin{array}{llllllll} \text{Lyap.} \Rightarrow & \text{unif.bnd on } \|Z\|_{BMO} & & & & & & \\ & \text{+sub-quadr.} & \Rightarrow & \text{sliceability} \Rightarrow & \text{exp.integrability} \Rightarrow & \text{solution.} \end{array}$$

From this presentation, we see exactly the motivations of each of the ideas involved in this logic process. We can also note that the passages "$\Rightarrow$" correspond to intuitive and straightforward proofs. Removing, for example, the node of exponential integrability (i.e. Theorem 2), it becomes immediately complicated to explain the last implication.

Thank you very much