

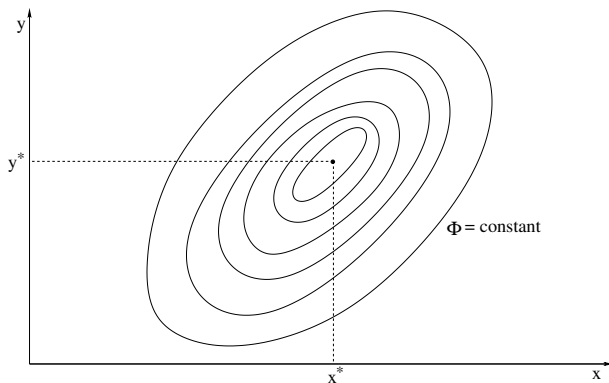
# Feedback Equilibrium Solutions to Noncooperative Differential Games

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# Optimal decision problem

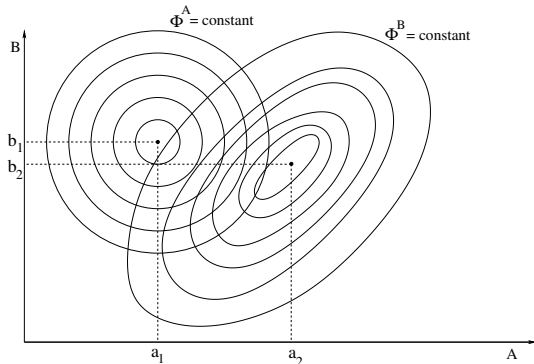
maximize:  $\Phi(x, y)$



The choice  $(x^*, y^*) \in \mathbb{R}^2$  yields the maximum payoff

# A game for two players

- Player A wishes to maximize his payoff  $\Phi^A(a, b)$
- Player B wishes to maximize his payoff  $\Phi^B(a, b)$



- Player A chooses the value of  $a \in A$
- Player B chooses the value of  $b \in B$

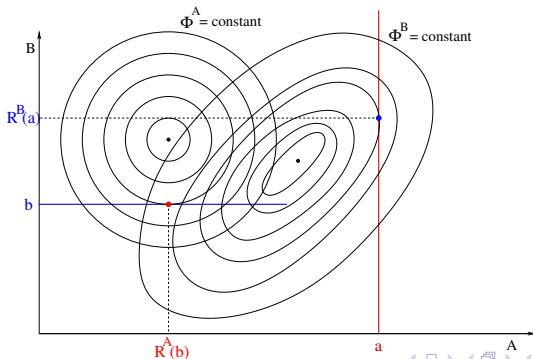
# The best reply map

If Player A adopts the strategy  $a$ , the set of best replies for Player B is

$$R^B(a) = \left\{ b; \Phi^B(a, b) = \max_{s \in B} \Phi^B(a, s) \right\}$$

If Player B adopts the strategy  $b$ , the set of best replies for Player A is

$$R^A(b) = \left\{ a; \Phi^A(a, b) = \max_{s \in A} \Phi^A(s, b) \right\}$$



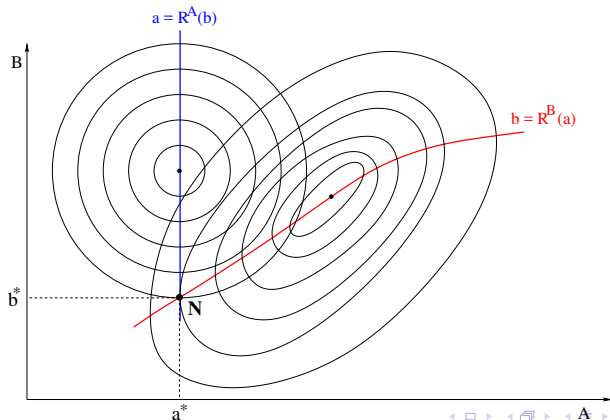
# Nash equilibrium solutions

A couple of strategies  $(a^*, b^*)$  is a **Nash equilibrium** if

$$a^* \in R^A(b^*)$$

and

$$b^* \in R^B(a^*)$$



# Optimal control problem

**Dynamics:**  $\dot{x}(t) = f(x(t), u(t)), \quad x(t_0) = x_0$

$x(t) \in \mathbb{R}^n =$  state of the system,  $u(\cdot) =$  control function

**Payoff:** 
$$J(u) \doteq \psi(x(T)) - \int_{t_0}^T L(x(t), u(t)) dt$$
$$= [\text{terminal payoff}] - [\text{running cost}]$$

**Value function:**  $V(t_0, x_0) =$  maximum payoff starting at  $(t_0, x_0)$

$$V_t + \max_{\omega} \left\{ \nabla_x V(t, x) \cdot f(x, \omega) - L(x, \omega) \right\} = 0 \quad (HJ)$$

$$V(T, x) = \psi(x)$$

# Differential game

**Dynamics:**  $\dot{x}(t) = f(x(t), u_1(t), u_2(t)), \quad x(t_0) = x_0$

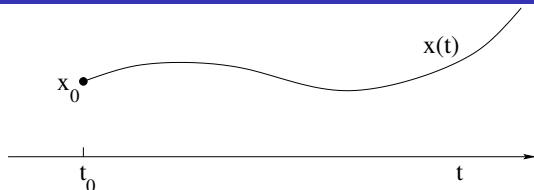
$x(t)$  = state of the system

$u_1(\cdot), u_2(\cdot)$  = controls implemented by the two players

**Payoffs:**  $J_i \doteq \psi_i(x(T)) - \int_{t_0}^T L_i(x(t), u_1(t), u_2(t)) dt, \quad i = 1, 2$

= [terminal payoff] - [running cost]

# Open-loop vs. Feedback controls



- For a control system, knowing the initial state  $x_0$  and the control  $u(\cdot)$  uniquely determine the trajectory

$$\dot{x}(t) = f(x(t), u(t)), \quad x(t_0) = x_0$$

- For a differential game, if a player knows the initial state and his own control, he still cannot recover the state  $x(t)$

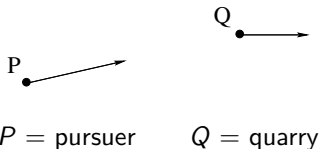
$$\dot{x}(t) = f(x(t), u_1(t), u_2(t)), \quad x(t_0) = x_0$$



For a differential game, the outcome can be very different, depending on the information that players have on the state of the system

In many cases, **feedback controls**  $u_i = u_i(t, x)$  are more relevant

### Pursuit game



Value function: minimum time need for  $P$  to catch  $Q$

# Nash equilibria

Seek: **feedback strategies**  $u_1^*(t, x)$ ,  $u_2^*(t, x)$  with the following properties

- Given the feedback control  $u_2 = u_2^*(t, x)$  adopted by the second player, for every initial data  $x(t_0) = x_0$  the assignment  $u_1 = u_1^*(t, x)$  provides a solution to the **optimal control problem for the first player**:

**Dynamics:**  $\dot{x} = f(x, u_1, u_2^*(t, x)), \quad x(t_0) = x_0$

**Payoff:**  $J_1 = \psi_1(x(T)) - \int_{t_0}^T L_1(x(t), u_1(t), u_2^*(t, x(t))) dt$

- Similarly, given the strategy  $u_1 = u_1^*(t, x)$  adopted by the first player, the feedback control  $u_2 = u_2^*(t, x)$  provides a solution to the optimal control problem for the second player.

# A system of H-J equations for the value functions

$V_1, V_2 =$  value functions for the two players

$$\begin{cases} V_{1,t} + \max_{\omega} \left\{ \nabla_x V_1(t, x) \cdot f(x, \omega, u_2^*) - L_1(x, \omega, u_2^*) \right\} = 0 \\ V_{2,t} + \max_{\omega} \left\{ \nabla_x V_2(t, x) \cdot f(x, u_1^*, \omega) - L_2(x, u_1^*, \omega) \right\} = 0 \end{cases} \quad (HJ)$$

**Terminal conditions:**

$$\begin{cases} V_1(T, x) = \psi_1(x) \\ V_2(T, x) = \psi_2(x) \end{cases}$$

$$u_1^*(t, x) = \operatorname{argmax}_{\omega} \left\{ \nabla_x V_1(t, x) \cdot f(x, \omega, u_2^*) - L_1(x, \omega, u_2^*) \right\}$$

$$u_2^*(t, x) = \operatorname{argmax}_{\omega} \left\{ \nabla_x V_2(t, x) \cdot f(x, u_1^*, \omega) - L_2(x, u_1^*, \omega) \right\}$$

system of first order PDEs, highly nonlinear !

# Explicit solutions: for Linear-Quadratic games

**Dynamics:**  $\dot{x}(t) = Ax(t) + \mathbf{b}_1 u_1(t) + \mathbf{b}_2 u_2(t) \quad x(t_0) = x_0$

**Payoffs:**  $J_i \doteq \mathbf{p}_i \cdot x(T) + x(T)^T Q_i x(T) - \int_{t_0}^T [x(t)^T R_i x(t) + |u_i(t)|^2] dt$

**Value functions:**

$$V_i(t, x) = x^T \Gamma_i(t) x + \beta_i(t) \cdot x + k_i(t) \quad i = 1, 2 \quad (*)$$

To find this solution, it suffices to determine the coefficients  $\Gamma_i(t)$ ,  $\beta_i(t)$ ,  $k_i(t)$ , by solving a system of ODEs

**Optimal controls:**  $u_i^*(t, x) = -(\beta_i(t) + 2x^T \Gamma_i(t)) \cdot \mathbf{b}_i$

T. Basar and G. Olsder, *Dynamic noncooperative game theory*. Academic Press, London, 1995.

D. L. Lukes, Equilibrium feedback control in linear games with quadratic costs. *SIAM J. Control Optim.* **9** (1971), 234–252.

A. J. Weeren, J. M. Schumacher, and J. Engwerda, Asymptotic analysis of linear feedback Nash equilibria in nonzero-sum linear-quadratic differential games. *J. Optim. Theory Appl.* **101** (1999), 693–723.

J. Engwerda, Feedback Nash equilibria in the scalar infinite horizon LQ-game *Automatica* **36** (2000), 135–139.

# Is the Linear-Quadratic approximation justified? Is the system (HJ) of nonlinear PDEs well posed?

Let  $V = (V_1, V_2)$  be a smooth solution, and let

$$V^\varepsilon(t, x) = V(t, x) + \varepsilon Z(t, x) + o(\varepsilon) \quad x \in \mathbb{R}^n$$

describe a small perturbation. How does  $Z$  behave?

$$\begin{pmatrix} Z_{1,t} \\ Z_{2,t} \end{pmatrix} + \sum_{\alpha=1}^n A^\alpha \begin{pmatrix} Z_{1,x_\alpha} \\ Z_{2,x_\alpha} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$A^\alpha = \begin{pmatrix} f_\alpha + \left( \nabla V_1 \cdot \frac{\partial f}{\partial u_2} - \frac{\partial L_1}{\partial u_2} \right) \frac{\partial u_2^*}{\partial V_{1,x_\alpha}} & \left( \nabla V_1 \cdot \frac{\partial f}{\partial u_2} - \frac{\partial L_1}{\partial u_2} \right) \frac{\partial u_2^*}{\partial V_{2,x_\alpha}} \\ \left( \nabla V_2 \cdot \frac{\partial f}{\partial u_1} - \frac{\partial L_2}{\partial u_1} \right) \frac{\partial u_1^*}{\partial V_{1,x_\alpha}} & f_\alpha + \left( \nabla V_2 \cdot \frac{\partial f}{\partial u_1} - \frac{\partial L_2}{\partial u_1} \right) \frac{\partial u_1^*}{\partial V_{2,x_\alpha}} \end{pmatrix}$$

# Hyperbolic systems

The Cauchy problem for the system with constant coefficients

$$\begin{pmatrix} Z_{1,t} \\ Z_{2,t} \end{pmatrix} + \sum_{\alpha=1}^n A^{\alpha} \begin{pmatrix} Z_{1,x_{\alpha}} \\ Z_{2,x_{\alpha}} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

is well posed only if the system is **hyperbolic**. That means:

*For every  $\xi = (\xi_1, \dots, \xi_n)$ , the matrix  $A(\xi) = \sum_{\alpha=1}^n A^{\alpha} \xi_{\alpha}$  has real eigenvalues.*

- Compute the solution by Fourier transform
- Stability in  $\mathbf{L}^2(\mathbb{R}^n)$   $\implies$  hyperbolicity

# An example

dynamics:  $\dot{x} = f(x, u_1, u_2) = f_0(x) + f_1(x)u_1 + f_2(x)u_2$

payoffs:  $J_i = \psi_i(x(T)) - \int_{t_0}^T \frac{1}{2} u_i^2(t) dt$

optimal feedback controls:  $u_i^*(t, x) = \nabla V_i(t, x) \cdot f_i(x)$

$$A(\xi) = \begin{pmatrix} f \cdot \xi & (\nabla V_1 \cdot f_2)(f_2 \cdot \xi) \\ (\nabla V_2 \cdot f_1)(f_1 \cdot \xi) & f \cdot \xi \end{pmatrix}$$

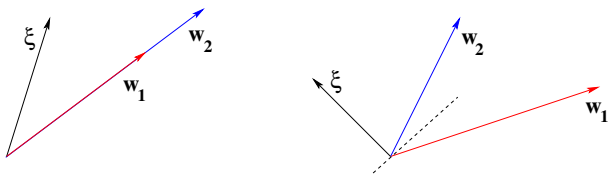
eigenvalues are real  $\iff (\nabla V_1 \cdot f_2)(f_2 \cdot \xi) (\nabla V_2 \cdot f_1)(f_1 \cdot \xi) \geq 0$



eigenvalues are real  $\iff (\mathbf{w}_1 \cdot \xi)(\mathbf{w}_2 \cdot \xi) \geq 0$

$$\mathbf{w}_1 = (\nabla V_2 \cdot f_1)f_1 \qquad \mathbf{w}_2 = (\nabla V_1 \cdot f_2)f_2$$

Cannot hold for all  $\xi$ , unless  $\mathbf{w}_1, \mathbf{w}_2$  are parallel, same orientation



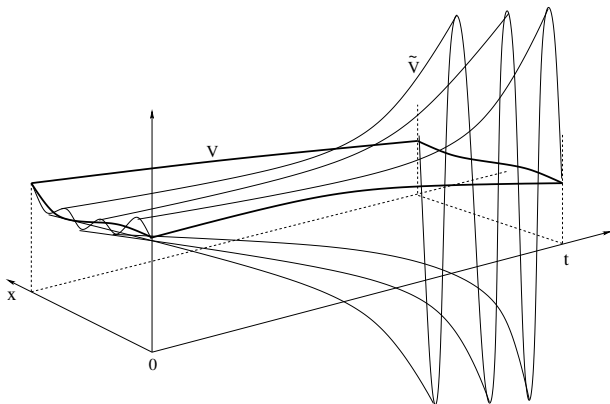
Possible in dimension 1, when  $\nabla V_1 \cdot \nabla V_2 \geq 0$

- In one space dimension, the Cauchy Problem can be well posed for a large set of data.
- In several space dimensions, generically the system is **not hyperbolic**, and the Cauchy Problem is **ill posed**

A.B., W.Shen, Small BV solutions of hyperbolic non-cooperative differential games, *SIAM J. Control Optim.* **43** (2004), 194–215.

A.B., W.Shen, Semi-cooperative strategies for differential games, *Intern. J. Game Theory* **32** (2004), 561–593.

A.B., Noncooperative differential games. *Milan J. Math.*, **79** (2011), 357–427.



For non-cooperative differential games in finite time horizon  
 linear - quadratic approximations are hardly justified !

What happens for games in infinite time horizon?

**Dynamics:**  $\dot{x} = f(x) + g_1(x)u_1 + g_2(x)u_2, \quad x(0) = x_0$

**cost functionals:**  $J_i \doteq \int_0^{+\infty} e^{-\gamma t} \left( \phi_i(x) + \frac{u_i^2}{2} \right) dt$

(running cost, exponentially discounted in time)

# A system of PDEs for the value functions

The value functions  $V_1, V_2$  for the two players satisfy the system of H-J equations

$$\begin{cases} \gamma V_1 &= (f \cdot \nabla V_1) - \frac{1}{2}(g_1 \cdot \nabla V_1)^2 - (g_2 \cdot \nabla V_1)(g_2 \cdot \nabla V_2) + \phi_1 \\ \gamma V_2 &= (f \cdot \nabla V_2) - \frac{1}{2}(g_2 \cdot \nabla V_2)^2 - (g_1 \cdot \nabla V_1)(g_1 \cdot \nabla V_2) + \phi_2 \end{cases}$$

Optimal feedback controls:  $u_i^*(x) = -\nabla V_i(x) \cdot g_i(x) \quad i = 1, 2$

nonlinear, implicit !

# Linear - Quadratic games

Assume that the dynamics is linear:

$$\dot{x} = (Ax + \mathbf{b}_0) + \mathbf{b}_1 u_1 + \mathbf{b}_2 u_2, \quad x(0) = y$$

and the cost functions are quadratic:

$$J_i = \int_0^{+\infty} e^{-\gamma t} \left( \mathbf{a}_i \cdot x + x^T P_i x + \frac{u_i^2}{2} \right) dt$$

Then the system of PDEs has a special solution of the form

$$V_i(x) = x^T \Gamma_i x + \beta_i \cdot x + k_i \quad i = 1, 2 \quad (*)$$

$$\text{optimal controls: } u_i^*(x) = -(\beta_i + 2x^T \Gamma_i) \cdot \mathbf{b}_i$$

To find this solution, it suffices to determine the coefficients  $k_i, \beta_i, \Gamma_i$  by solving a system of algebraic equations

# Validity of linear-quadratic approximations ?

Assume the dynamics is almost linear

$$\dot{x} = f_0(x) + g_1(x)u_1 + g_2(x)u_2 \approx (Ax + \mathbf{b}_0) + \mathbf{b}_1u_1 + \mathbf{b}_2u_2, \quad x(0) = y$$

and the cost functions are almost quadratic

$$J_i = \int_0^{+\infty} e^{-\gamma t} \left( \phi_i(x) + \frac{u_i^2}{2} \right) dt \approx \int_0^{+\infty} e^{-\gamma t} \left( \mathbf{a}_i \cdot x + x^T P_i x + \frac{u_i^2}{2} \right) dt$$

Is it true that the nonlinear game has a feedback solution close to the linear-quadratic game?

$$\dot{x} = (a_0x + b_0) + b_1u_1 + b_2u_2$$

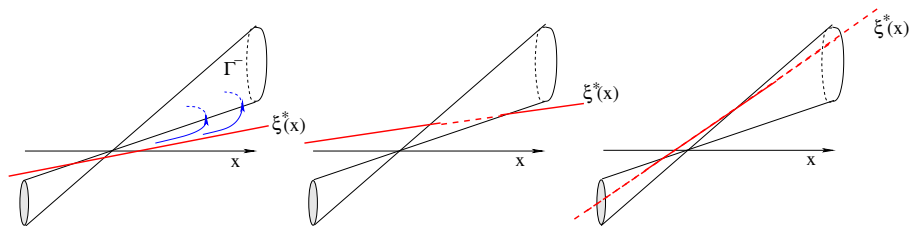
The ODE for the derivatives of the value functions  $\xi_i = V'_i$  takes the form

$$\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} \xi'_1 \\ \xi'_2 \end{pmatrix} = \begin{pmatrix} \psi_1(x, \xi_1, \xi_2) \\ \psi_2(x, \xi_1, \xi_2) \end{pmatrix}, \quad A_{ij} = A_{ij}(x, \xi_1, \xi_2)$$

- The map  $(x, \xi_1, \xi_2) \mapsto \det A(x, \xi_1, \xi_2)$  is a homogeneous quadratic polynomial
- An affine solution exists:  $\xi_1^*(x) = k_1x + \beta_1$ ,  $\xi_2^*(x) = k_2x + \beta_2$
- The map  $x \mapsto \det A(x, \xi_1^*(x), \xi_2^*(x))$  is a quadratic polynomial



Three main cases:

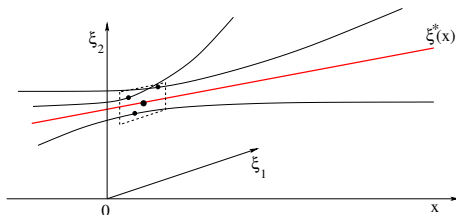


$$\Gamma^- = \left\{ (x, \xi_1, \xi_2); \quad \det A(x, \xi_1, \xi_2) \leq 0 \right\}$$

Case 1:  $\det A(x, \xi_1^*(x), \xi_2^*(x)) \neq 0$  for all  $x \in \mathbb{R}$

$$\begin{pmatrix} \xi_1' \\ \xi_2' \end{pmatrix} = A^{-1}(x, \xi_1, \xi_2) \begin{pmatrix} \psi_1(x, \xi_1, \xi_2) \\ \psi_2(x, \xi_1, \xi_2) \end{pmatrix}$$

- The linear-quadratic game has a **2-parameter family of Nash equilibrium solutions** in feedback form. One is affine, the other are nonlinear.
- All of the above solutions are stable w.r.t. small nonlinear perturbations of the dynamics and the cost functions.



## Case 2: $\det A$ vanishes at two points $\bar{x}_1 < \bar{x}_2$

$$\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} \xi'_1 \\ \xi'_2 \end{pmatrix} = \begin{pmatrix} \psi_1(x, \xi_1, \xi_2) \\ \psi_2(x, \xi_1, \xi_2) \end{pmatrix}$$

Equivalent system: 
$$\begin{cases} A_{11} d\xi_1 + A_{12} d\xi_2 - \psi_1 dx = 0 \\ A_{21} d\xi_1 + A_{22} d\xi_2 - \psi_2 dx = 0 \end{cases}$$

Setting: 
$$\mathbf{v} \doteq \begin{pmatrix} -\psi_1 \\ A_{11} \\ A_{12} \end{pmatrix}, \quad \mathbf{w} \doteq \begin{pmatrix} -\psi_2 \\ A_{21} \\ A_{22} \end{pmatrix},$$

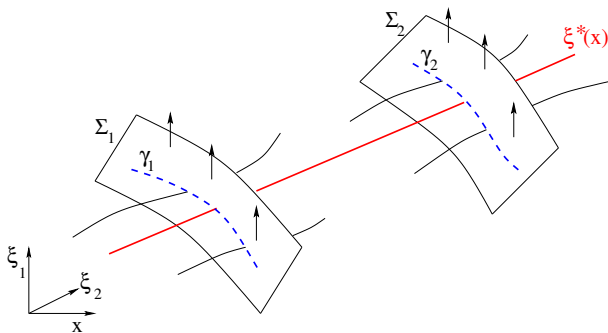
We seek continuously differentiable functions  $x \mapsto (\xi_1(x), \xi_2(x))$  whose graph is obtained by concatenating trajectories of the system

$$\begin{pmatrix} \dot{x} \\ \dot{\xi}_1 \\ \dot{\xi}_2 \end{pmatrix} = \mathbf{v} \times \mathbf{w} = \begin{pmatrix} A_{11}A_{22} - A_{12}A_{21} \\ A_{22}\psi_1 - A_{12}\psi_2 \\ A_{11}\psi_2 - A_{21}\psi_1 \end{pmatrix}.$$

$$\det \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} = 0 \quad \text{on} \quad \Sigma_1 \cup \Sigma_2$$

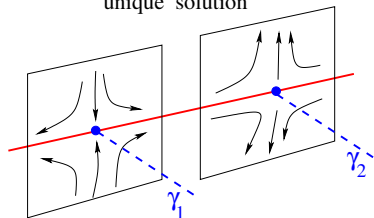
Under generic conditions on the coefficients of the linear-quadratic problem, there exists two curves  $\gamma_1 \subset \Sigma_1$ ,  $\gamma_2 \subset \Sigma_2$  such that

- $\mathbf{v} \times \mathbf{w} = 0$  on  $\gamma_1$  and on  $\gamma_2$
- $\mathbf{v} \times \mathbf{w}$  is vertical on  $\Sigma_1 \setminus \gamma_1$  and on  $\Sigma_2 \setminus \gamma_2$

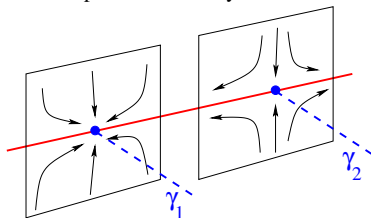


# Three generic cases

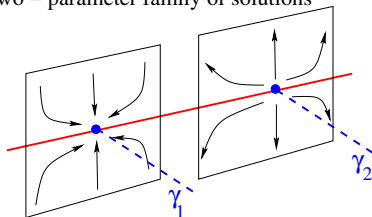
unique solution



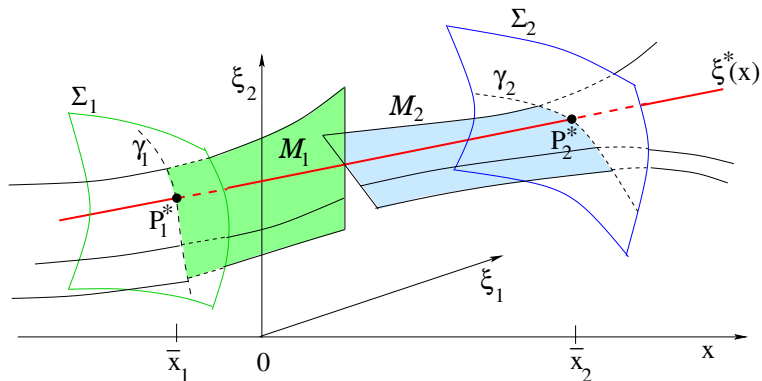
one - parameter family of solutions



two - parameter family of solutions



# The saddle-saddle case



Under generic assumptions, a unique solution exists

# Stability under perturbations, on the entire real line

(A.B., K.Nguyen, 2016)

$$\dot{x} = a_0x + f_0(x) + (b_1 + h_1(x))u_1 + (b_2 + h_2(x))u_2$$

$$J_i = \int_0^{+\infty} e^{-\gamma t} \left( \mathbf{a}_i x + P_i x^2 + \eta_i(x) + \frac{u_i^2}{2} \right) dt$$

Under generic assumptions on the coefficients  $a_0, b_1, b_2, \dots$ , for any  $\varepsilon > 0$  there exists  $\delta > 0$  such that the following holds. If the perturbations satisfy

$$\|f_0\|_{C^2} + \|\eta_1\|_{C^2} + \|\eta_2\|_{C^2} + \|h_1\|_{C^1} + \|h_2\|_{C^1} \leq \delta,$$

then the perturbed equations for  $\xi_1 = V'_1$ ,  $\xi_2 = V'_2$  admit a solution such that

$$|\xi_1(x) - \xi_1^*(x)| + |\xi_2(x) - \xi_2^*(x)| \leq \varepsilon(1 + |x|) \quad \text{for all } x \in \mathbb{R}$$

# Alternative models: 1 - a stochastic evolution equation

**Dynamics:**  $dx = f(x, u_1, u_2) dt + \sigma dW_t, \quad x(t_0) = x_0 \in \mathbb{R}^n$

$$\sigma = \sigma(x), \quad \text{Rank}(\sigma\sigma^T) = n$$

$W =$  Brownian motion

**Payoffs:**  $J_i \doteq E \left[ \psi_i(x(T)) - \int_{t_0}^T L_i(x(t), u_1(t), u_2(t)) dt \right], \quad i = 1, 2$



# A parabolic system for the value functions $V_1, V_2$

$$\begin{cases} V_{1,t} + \max_{\omega} \left\{ \nabla_x V_1(t, x) \cdot f(x, \omega, u_2^*) - L_1(x, \omega, u_2^*) \right\} + \frac{1}{2} \sum_{jk} a_{jk} V_{1,x_j x_k} = 0 \\ V_{2,t} + \max_{\omega} \left\{ \nabla_x V_2(t, x) \cdot f(x, u_1^*, \omega) - L_2(x, u_1^*, \omega) \right\} + \frac{1}{2} \sum_{jk} a_{jk} V_{2,x_j x_k} = 0 \end{cases}$$

$$\sigma \sigma^T = (a_{jk}), \quad \text{Terminal conditions:} \quad \begin{cases} V_1(T, x) = \psi_1(x) \\ V_2(T, x) = \psi_2(x) \end{cases}$$

$$u_1^*(t, x) = \operatorname{argmax}_{\omega} \left\{ \nabla_x V_1(t, x) \cdot f(x, \omega, u_2^*) - L_1(x, \omega, u_2^*) \right\}$$

$$u_2^*(t, x) = \operatorname{argmax}_{\omega} \left\{ \nabla_x V_2(t, x) \cdot f(x, u_1^*, \omega) - L_2(x, u_1^*, \omega) \right\}$$

parabolic system  $\implies$  backward Cauchy problem is well posed

## 2 - Zero-sum games

$$[\text{payoff for Player 1}] + [\text{payoff for Player 2}] = 0$$

$\Rightarrow$  the value function  $V = V_1 = -V_2$  satisfies a scalar H-J equation

$$V_t + H(x, V, \nabla V) = 0, \quad V(T, x) = \psi(x)$$

$\Rightarrow$  a unique *viscosity solution* exists

M. Bardi and I. Capuzzo Dolcetta, *Optimal control and viscosity solutions of Hamilton- Jacobi-Bellman equations*, Birkhäuser, 1997.

# The cooperative - competitive solution

**Dynamics:**  $\dot{x} = f(x, u_1, u_2) \quad x(t_0) = x_0$

**Payoffs:**  $J_i \doteq \psi_i(x(T)) - \int_{t_0}^T L_i(x(t), u_1(t), u_2(t)) dt$

- Cooperatively choose  $u_1, u_2$  in order to maximize the sum  $J_1 + J_2$
- Add a side payment from one player to the other, determined by solving a zero-sum game

A. Kalai and E. Kalai, Cooperation in Strategic Games Revisited  
*Quarterly J. Economics* **128** (2013), 917–966.

### Cooperative control problem:

$$\max_{u_1, u_2} J_1 + J_2 \doteq (\psi_1 + \psi_2)(x(T)) - \int_{t_0}^T (L_1 + L_2)(x(t), u_1(t), u_2(t)) dt$$

subject to:  $\dot{x} = f(x, u_1, u_2) \quad x(t_0) = x_0$

Maximum combined payoff:  $V(x_0)$

### Competitive zero-sum game:

$$J_1 - J_2 \doteq (\psi_1 - \psi_2)(x(T)) - \int_{t_0}^T (L_1 - L_2)(x(t), u_1(t), u_2(t)) dt$$

Payoff for Player 1:  $W(x_0)$

Co-co values of the game (achieved by a side payment):

$$\text{Player 1 receives } \frac{V + W}{2}, \quad \text{Player 2 receives } \frac{V - W}{2}$$

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