Deterministic and stochastic 2D Euler equations with random initial conditions

Franco Flandoli, Scuola Normale Superiore di Pisa

Stochastic Modeling, Verona, 2017

Random initial conditions in ODE and PDE

We all know that the Cauchy problem

$$x' = f(x), \qquad x|_{t=0} = x_0$$

has a unique solution when f il Lipschitz continuous.
Below this regularity of f there are various counterexamples. Is existence and uniqueness still true, however, for "most" initial conditions?
One way to characterize "large" classes of initial conditions is by means of *Probability*: we look for existence and uniqueness *for almost every initial condition* with respect to some probability measure.
This is the motivation, in this talk, to introduce *random initial conditions*:

improve the well-posedness theory.

Several directions

- special examples of ODEs (interacting particle systems; interacting point vortices)
- general classes of ODEs (weakly differentiable drift)
- general classes of ODEs in infinite dimensional spaces
- special examples of PDEs (dispersive equations, 2D Euler equations).

Interacting particle systems

Oscar E. Lanford III, Time evolution of large particle systems, 1975, 1-111

Difficulty: too many particles (hence **energy blow-up**) in bounded regions. Lanford proves that this cannot happen, for a.e. initial condition.

$$\frac{dq_i}{dt} = p_i, \qquad \frac{dp_i}{dt} = -\sum_{j \neq i} \nabla U(q_i - q_j)$$

with initial conditions q_i^0, p_i^0

$$H(q,p) = \frac{1}{2}\sum_i p_i^2 + \sum_{i,j,i \neq j} U(q_i - q_j)$$

$$\mu_\beta(dq, dp) = \frac{1}{Z_\beta} e^{-\beta H(q,p)} dq dp$$

In the case of finitely many particles, good potential U, $\beta > 0$: the Gibbs measure μ_β is well defined and invariant.

How to use invariance

Assume we want to prove that particles, initially distributed in uniform way, cannot move too fast and concentrate too much. Consider the quantity

$$\sup_{t\in[0,T]} \left| q_i(t) - q_i^0 \right| .$$

We want to prove it cannot be too large. More precisely, consider

$$\sum_i \sup_{t\in[0,T]} \left| q_i(t) - q_i^0 \right|^2 .$$

We have

$$\sum_i \sup_{t\in[0,T]} \left| q_i(t) - q_i^0 \right|^2 \leq \sqrt{T} \int_0^T \sum_i \left| p_i(s) \right|^2 ds.$$

(Typical strategy: *controlling quantities of interest by time integrals*.)

How to use invariance

Then integrate with respect to $\mu_\beta\left(dq^0, dp^0\right)$:

$$\begin{aligned} &\int \left(\sum_i \sup_{t\in[0,T]} \left| q_i(t) - q_i^0 \right|^2 \right) \mu_\beta\left(dq^0, dp^0\right) \\ \leq\ & \sqrt{T} \int_0^T \int \left(\sum_i \left| p_i(s) \right|^2 \right) \mu_\beta\left(dq^0, dp^0\right) ds \\ =\ & T^{3/2} \int \sum_i \left| p_i^0 \right|^2 \mu_\beta\left(dq^0, dp^0\right) \leq C_T \end{aligned}$$

$$\Longrightarrow \sum_i \sup_{t\in[0,T]} \left| q_i(t) - q_i^0 \right|^2 < C\left(q^0, p^0\right) \qquad \text{for } \mu_\beta\text{-a.e. } \left(q^0, p^0\right).$$

(We have *interchanged integrations, used invariance and integrability of* $\sum_i \left| p_i^0 \right|^2$.)

Interacting point vortices

C. Marchioro and M. Pulvirenti, Mathematical theory of incompressible nonviscous fluids, 1994 (Chapter 4)

Point vortices. Difficulty: **singular interaction kernel**, blow-up if vortices meet each other:

$$\frac{dx_i}{dt} = \sum_{j \neq i} \frac{\left(x_i - x_j\right)^{\perp}}{\left|x_i - x_j\right|^2}$$

There are explicit examples of initial conditions that lead to coalescence. They prove it cannot happen, for a.e. initial condition with respect to Lebesgue measure.
Estimate

$$\sup_{t \in [0,T]} \sum_{i,j} \log_{+} \left|x_i\left(t\right) - x_j\left(t\right)\right|$$

by the time integral of an integrable function; and use invariance of Lebesgue measure.

R. i. conditions in ODEs, finite and infinite dimensional, and related continuity equation

R.J.Di Perna and P.L. Lions, P.L., Invent. Math. 1989; L. Ambrosio, Invent. Math. 2004

$$x'(t) = f(t, x(t)) \qquad \text{in } \mathbb{R}^d$$

when f is only weakly differentiable, like $f \in W^{1,1}$. They construct "Lagrangian flows" (the initial condition is random).
Variuous extensions to infinite dimensional spaces, Wiener space:
(after A. B. Cruzeiro, JFA 1984 and V. Bogachev, E. M. Wolf, JFA 1999)
L. Ambrosio, A. Figalli, JFA 2009,
S. Fang, D. Luo, BSM 2010,
L. Ambrosio, D. Trevisan, APDE 2014,
G. Da Prato, F. Flandoli and M. Röckner, SPDE 2014,
A. V. Kolesnikov, M. Röckner, JFA 2014

Random initial conditions in dispersive PDE

J. Bourgain, Nonlinear Schrodinger equation, 1996
N. Burq, N. Tzvetkov, Wave equations 2008
and many others (T. Oh, N. Visciglia, G. Richards, ...)

$$\partial_{tt}^2 u = \Delta u - u^3 \qquad \text{in } \mathbb{T}^3$$

$$(u, \partial_t u) \in \mathcal{H}^s = H^s \times H^{s-1}$$

- $s \geq 1$: well posed by energy methods
- $s \in \left(\frac{1}{2}, 1\right)$: well posed by Strichartz estimates
- $s \in \left[0, \frac{1}{2}\right]$: **well posed for a.e. initial condition** with respect to certain Gaussian measures.

The 2D Euler equations

- Review of results on deterministic 2D Euler equations
- Albeverio-Cruzeiro in the framework of more classical results
- White noise initial conditions
- Weak vorticity formulation
- Point vortex approximation
- Main results and perspectives

The 2D Euler equations

Let us consider the equations on the torus $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$.
Euler equations for $(u, p) =$ (velocity,pressure) read

$$\begin{aligned} \partial_t u + u \cdot \nabla u + \nabla p &= 0 \\ \operatorname{div} u &= 0. \end{aligned}$$

The vorticity $\omega = \partial_2 u_1 - \partial_1 u_2$ satisfies

$$\partial_t \omega + u \cdot \nabla \omega = 0.$$

We shall always consider the vorticity formulation.
Main formal invariants (among others):

$$\text{kinetic energy} = \frac{1}{2} \int_{\mathbb{T}^2} |u(x)|^2 \, dx$$

$$\text{enstrophy} = \int_{\mathbb{T}^2} \omega(x)^2 \, dx.$$

Review of results on 2D Euler equations

- existence and uniqueness, when $\omega_0 \in L^\infty$ (Wolibner, Yudovich)
- existence for $\omega_0 \in L^p$, $p \geq 2$ (uniqueness is open) (velocity $u \in W^{1,p}$)
- existence for $\sim$positive measures $\omega_0(dx)$ of class H^{-1} (Delort) (velocity $u \in L^2$)
- existence and uniqueness for a.e. point vortex measure $\omega_0(dx) = \sum_{i=1}^N \omega_i \delta_{X_0^i}$, which belongs to $H^{-1-} := \bigcap_{\epsilon>0} H^{-1-\epsilon}$ (Marchioro-Pulvirenti) (velocity $u \notin L^2$)
- **existence for μ-a.e.** $\omega_0 \in H^{-1-}$ (μ described below) (Albeverio-Cruzeiro, CMP '90)

New results

1. prove existence for μ-a.e. $\omega_0 \in H^{-1-}$ (suitable μ) using a new approach (called *weak vorticity formulation*)
2. prove that such solutions are *limit of point vortices and also of L^∞-solutions*.

White noise initial conditions

We assume that ω_0 is a *white noise* on $\mathbb{T}^2$.
White noise on $\mathbb{T}^2$ is a distributional-valued stochastic process $\omega_0 : \Xi \to C^\infty\left(\mathbb{T}^2\right)'$ (here $(\Xi, \mathcal{F}, P)$ is a probability space) such that

$$\mathbb{E}\left[\langle\omega_0, \phi\rangle \langle\omega_0, \psi\rangle\right] = \langle\phi, \psi\rangle$$

for all $\phi, \psi \in C^\infty\left(\mathbb{T}^2\right)$. In more heuristic terms,

$$\mathbb{E}\left[\omega_0\left(x\right)\omega_0\left(y\right)\right] = \delta\left(x - y\right).$$

It will turn out that the solutions constructed below is a white noise *at every time* (similarly to KDV equations, and to stochastic Burgers equation of KPZ theory).

The enstrophy measure

The law of white noise on $\mathbb{T}^2$, in this fluid dynamic framework, is called *enstrophy measure*, since it is heuristically given by

$$\mu(d\omega) = \frac{1}{Z} \exp\left(-\frac{1}{2} \int_{\mathbb{T}^2} \omega^2 dx \right) d\omega.$$

This measure is supported on $H^{-1-}(\mathbb{T}^2)$ but

$$\begin{aligned} \mu\left(H^{-1}(\mathbb{T}^2)\right) &= 0 \\ \mu\left(\mathcal{M}(\mathbb{T}^2)\right) &= 0. \end{aligned}$$

Weak vorticity formulation

We need to give a meaning to the nonlinear term of the equation

$$\partial_t \omega + u \cdot \nabla \omega = 0$$

when ω is a white noise.

Remark. There is an analogy with KPZ theory, but in fact here it is much easier.

Trivial integration by parts on test functions $\phi \in C^\infty\left(\mathbb{T}^2\right)$

$$\int_{\mathbb{T}^2} \omega\left(x\right) u\left(x\right) \cdot \nabla \phi\left(x\right) dx \qquad \text{(formal notation)}$$

is not sufficient, since u is not regular enough (u is not even L^2).

Weak vorticity formulation

First, using Biot-Savart formula $u(x) = \int_{\mathbb{T}^2} K(x-y)\,\omega(y)\,dy$ (where $|K(x)| \sim \frac{1}{|x|}$ near $x = 0$) we rewrite

$$\int_{\mathbb{T}^2} \omega(x)\,u(x) \cdot \nabla\phi(x)\,dx = \int_{\mathbb{T}^2}\int_{\mathbb{T}^2} K(x-y) \cdot \nabla\phi(x)\,\omega(x)\,\omega(y)\,dxdy.$$

The function $K(x-y) \cdot \nabla\phi(x)$ **is singular at** $x = y$. Then we symmetrize:

$$\begin{aligned} &= \frac{1}{2}\int_{\mathbb{T}^2}\int_{\mathbb{T}^2} K(x-y)\left(\nabla\phi(x) - \nabla\phi(y)\right)\omega(x)\,\omega(y)\,dxdy \\ &=: \int_{\mathbb{T}^2}\int_{\mathbb{T}^2} H_\phi(x,y)\,\omega(x)\,\omega(y)\,dxdy. \end{aligned}$$

The function $H_\phi(x,y)$ **is bounded, smooth outside the diagonal**, discontinuous along the diagonal. Can we integrate $H_\phi(x,y)$ against $\omega(x)\,\omega(y)\,dxdy$?

Weak formulation of the Euler equations for white noise

Assume $\omega : \Xi \to C^\infty\left(\mathbb{T}^2\right)'$ is a white noise. Can we give a meaning to

$$\int_{\mathbb{T}^2}\int_{\mathbb{T}^2} H_\phi\left(x,y\right)\omega\left(x\right)\omega\left(y\right)dxdy?$$

Being $\omega \in H^{-1-}\left(\mathbb{T}^2\right)$, we have at least

$$\omega\otimes\omega \in H^{-2-}\left(\mathbb{T}^2\times\mathbb{T}^2\right) \text{ with probability one}$$

hence $\langle\omega\otimes\omega, f\rangle$ is well defined for $f \in H^{2+}\left(\mathbb{T}^2\times\mathbb{T}^2\right)$. The question is: can we define

$$\left\langle\omega\otimes\omega, H_\phi\right\rangle$$

in spite of the fact that H_ϕ is not of class $H^{2+}\left(\mathbb{T}^2\times\mathbb{T}^2\right)$?

Main lemma

Lemma

If $\omega : \Xi \to C^{\infty}\left(\mathbb{T}^2\right)'$ *is a white noise and* $f \in H^{2+}\left(\mathbb{T}^2 \times \mathbb{T}^2\right)$ *is symmetric, then* $\int f(x,x)\,dx = \mathbb{E}\left[\langle \omega \otimes \omega, f\rangle\right]$ *and*

$$\mathbb{E}\left[\left|\langle \omega \otimes \omega, f\rangle - \int f(x,x)\,dx\right|^2\right] = 2\int\int f^2(x,y)\,dxdy.$$

Let us see the formal proof (becomes rigorous by smoothing the WN)

$$\begin{aligned}&\mathbb{E}\left[\left|\int\int f(x,y)\,\omega(x)\,\omega(y)\,dxdy\right|^2\right]\\ =\;&\int\int\int\int f(x,y)\,f\left(x',y'\right)\mathbb{E}\left[\omega(x)\,\omega(y)\,\omega\left(x'\right)\omega\left(y'\right)\right]dxdydx'dy'\end{aligned}$$

Proof of the main lemma

$$
\begin{aligned}
& E\left[\omega(x)\,\omega(y)\,\omega\left(x'\right)\omega\left(y'\right)\right] \\
= & \ \delta(x-y)\,\delta\left(x'-y'\right)+\delta\left(x-x'\right)\delta\left(y-y'\right)+\delta\left(x-y'\right)\delta\left(x'-y\right)
\end{aligned}
$$

by Gaussian rules for moments (Isserlis-Wick theorem). Hence

$$
\begin{aligned}
& \int\int\int\int f(x,y)\,f\left(x',y'\right)\mathbb{E}\left[\omega(x)\,\omega(y)\,\omega\left(x'\right)\omega\left(y'\right)\right]dxdydx'dy' \\
= & \int\int\int\int f(x,y)\,f\left(x',y'\right)\delta(x-y)\,\delta\left(x'-y'\right)dxdydx'dy'+\ldots \\
= & \int\int f(x,x)\,f\left(x',x'\right)dxdx'+\ldots
\end{aligned}
$$

and the proof becomes a simple computation.

Consequence of the main lemma

Theorem

Let $\omega : \Xi \to C^{\infty}\left(\mathbb{T}^2\right)'$ *be a white noise and* $\phi \in C^{\infty}\left(\mathbb{T}^2\right)$ *be given. Assume that* $H_{\phi}^{n} \in H^{2+}\left(\mathbb{T}^2 \times \mathbb{T}^2\right)$ *are symmetric and approximate* H_{ϕ} *in the following sense:*

$$\begin{aligned} \lim_{n\to\infty} \int \int \left(H_{\phi}^{n} - H_{\phi}\right)^2 (x, y)\, dxdy &= 0 \\ \lim_{n\to\infty} \int H_{\phi}^{n}(x, x)\, dx &= 0. \end{aligned}$$

Then the sequence of processes $s \mapsto \left\langle \omega_s \otimes \omega_s, H_{\phi}^{n} \right\rangle$ *is Cauchy in* $L^2\left(\Omega; L^2(0, T)\right)$*. We denote its limit by*

$$s \mapsto \left\langle \omega_s \otimes \omega_s, H_{\phi} \right\rangle.$$

The limit is the same when $\lim_{n\to\infty} \int \int \left(H_{\phi}^{n} - \widetilde{H}_{\phi}^{n}\right)^2 (x, y)\, dxdy = 0$.

Definition of solution

Definition

We say that a measurable map $\omega_{\cdot} : \Xi \times [0, T] \to C^{\infty}\left(\mathbb{T}^2\right)'$ is a **white noise solution** of Euler equations if ω_t is a white noise at every time $t \in [0, T]$, the paths are of class $C\left([0, T]; H^{-1-}\left(\mathbb{T}^2\right)\right)$ and, for every $\phi \in C^{\infty}\left(\mathbb{T}^2\right)$, we have the identity a.s.

$$\langle \omega_t, \phi \rangle = \langle \omega_0, \phi \rangle + \int_0^t \left\langle \omega_s \otimes \omega_s, H_\phi \right\rangle ds.$$

Random point vortex dynamics

Consider, for every $N \in \mathbb{N}$, the finite dimensional dynamics in $\left(\mathbb{T}^2\right)^N$

$$\frac{dX_t^{i,N}}{dt} = \sum_{j=1}^{N} \frac{1}{\sqrt{N}} \xi_j K\left(X_t^{i,N} - X_t^{j,N}\right) \qquad i = 1, ..., N$$

with initial condition $\left(X_0^{1,N}, ..., X_0^{N,N}\right) \in \left(\mathbb{T}^2\right)^N$, where K is Biot-Savard kernel on $\mathbb{T}^2$, with $K(0) := 0$ to neglect self-interaction.

Theorem (Marchioro-Pulvirenti)

*Given $\xi_1, \dots, \xi_N$, for $\otimes_N Leb_{\mathbb{T}^2}$- **almost every** $\left(X_0^{1,N}, ..., X_0^{N,N}\right) \in \Delta_N^c$, **there is a unique solution** $\left(X_t^{1,N}, ..., X_t^{N,N}\right)$ with the property that $\left(X_t^{1,N}, ..., X_t^{N,N}\right) \in \Delta_N^c$ for all $t \geq 0$.*

Here

$$\Delta_N = \left\{\left(x^1, ..., x^N\right) \in \left(\mathbb{T}^2\right)^N : x^i = x^j \text{ for some } i \neq j,\ i, j = 1, ..., n\right\}.$$

Random point vortex dynamics

Assume $\xi_1, \ldots, \xi_N$ are random intensities, distributed as $N(0,1)$, $X_0^{1,N}, \ldots, X_0^{N,N}$ are random and uniformly distributed, all independent of each other. Consider the **measure-valued vorticity** field

$$\omega_0^N = \frac{1}{\sqrt{N}} \sum_{n=1}^{N} \xi_n \delta_{X_0^n}.$$

We have $\mathbb{E}\left[\omega_0^N\right] = 0$ and

$$\langle Q_N \varphi, \psi \rangle := \mathbb{E}\left[\left\langle \omega_0^N, \varphi \right\rangle \left\langle \omega_0^N, \psi \right\rangle\right] = \int_{\mathbb{T}^2} \varphi(x) \psi(x) \, dx$$

for all $\varphi, \psi \in C^\infty\left(\mathbb{T}^2\right)$. (as for white noise). One can prove that

$$\omega_0^N \overset{Law}{\rightharpoonup} \omega_{WN}$$

in $H^{-1-\delta}$ for every $\delta > 0$.

Theorem

Consider the vortex dynamics with random intensities $(\xi_1, \ldots, \xi_N)$ *and random initial positions* $\left(X_0^1, \ldots, X_0^N\right)$ *as above. For a.e. value of* $\left(\xi_1, \ldots, \xi_N, X_0^1, \ldots, X_0^N\right)$ *the dynamics* $\left(X_t^{1,N}, \ldots, X_t^{N,N}\right)$ *is well defined in* Δ_N^c *for all* $t \geq 0$*, and the associated measure-valued vorticity* ω_t^N *satisfies the weak vorticity formulation. The stochastic process* ω_t^N *is stationary in time and space-homogeneous.*

Lemma

for all $f : \mathbb{T}^2 \times \mathbb{T}^2 \to \mathbb{R}$ *symmetric, bounded and measurable*

$$\mathbb{E}\left[\left\langle \omega_t^N \otimes \omega_t^N, f \right\rangle^2\right] = \frac{3}{N} \int f^2(x,x)\, dx + \left(\int f(x,x)\, dx\right)^2 + 2 \int \int f^2(x,y)\, dxdy.$$

Albeverio-Cruzeiro revisited

Theorem

i) **There exists** *a probability space* $(\Xi, \mathcal{F}, P)$ *and a measurable map* $\omega_\cdot : \Xi \times [0, T] \to C^\infty (\mathbb{T}^2)'$ *such that* $\omega_\cdot$ *is a time-stationary white noise solution of Euler equations.*

ii) The **random point vortex system** *has a subsequance which* **converges** *in law to this solution.*

Main steps in the proof

Let Q^N be the law of ω^N on

$$\mathcal{X} := C\left([0, T]; H^{-1-\delta}\right). \tag{1}$$

We prove that the family $\left\{Q^N\right\}_{N\in\mathbb{N}}$ is tight in $\mathcal{X}$ by proving it is bounded in probability in

$$\mathcal{Y} := L^p\left(0, T; H^{-1-\delta/2}\left(\mathbb{T}^2\right)\right) \cap W^{1,2}\left(0, T; H^{-4}\left(\mathbb{T}^2\right)\right)$$

for large p ($\mathcal{Y} \subset \mathcal{X}$ is compact by Simon).

Compactness in space

From stationarity of ω_t^N

$$\mathbb{E}\left[\int_0^T \left\|\omega_t^N\right\|_{H^{-1-\delta/2}}^p dt\right] = \int_0^T \mathbb{E}\left[\left\|\omega_t^N\right\|_{H^{-1-\delta/2}}^p\right] dt = T\mathbb{E}\left[\left\|\omega_0^N\right\|_{H^{-1-\delta/2}}^p\right]$$

$$= T\mathbb{E}\left[\left\|\frac{1}{\sqrt{N}}\sum_{n=1}^N \xi_n \delta_{X_0^n}\right\|_{H^{-1-\delta/2}}^p\right] \leq C_p T.$$

Hence the family $\left\{Q^N\right\}_{N\in\mathbb{N}}$ is bounded in probability in $L^p\left(0, T; H^{-1-\delta/2}\left(\mathbb{T}^2\right)\right)$.

Compactness in time

We use the equation in its weak vorticity formulation.
For all $\phi \in C^\infty(\mathbb{T}^2)$, $\partial_t \langle \omega_t^N, \phi \rangle = \langle \omega_t^N \otimes \omega_t^N, H_\phi \rangle$, hence

$$\begin{aligned}\mathbb{E}\left[\left|\partial_t \left\langle \omega_t^N, \phi \right\rangle\right|^2\right] &= \mathbb{E}\left[\left|\left\langle \omega_t^N \otimes \omega_t^N, H_\phi \right\rangle\right|^2\right] \\ &\leq C \left\| H_\phi \right\|_\infty^2 \leq C \left\| D^2 \phi \right\|_\infty^2 .\end{aligned}$$

With $\phi = e_k$ we get

$$\mathbb{E}\left[\left|\partial_t \left\langle \omega_t^N, e_k \right\rangle\right|^2\right] \leq C \left| k \right|^4$$

$$\mathbb{E}\left[\int_0^T \left\| \partial_t \omega_t^N \right\|_{H^{-\gamma'}}^2 dt\right] \leq C \mathbb{E}\left[\int_0^T \sum_k \left(1 + |k|^2\right)^{-\gamma'} |k|^4 \, dt\right] < \infty$$

for $\gamma' > 3$. The family $\left\{Q^N\right\}_{N \in \mathbb{N}}$ is bounded in probability in $W^{1,2}\left(0, T; H^{-\gamma'}(\mathbb{T}^2)\right)$.

Passage to the limit

From Prohorov theorem, there exists $\left\{Q^{N_k}\right\}_{k\in\mathbb{N}}$ which converges weakly, in X, to a probability measure Q.
A process $\omega.$ with law Q is time-stationary and ω_t is white noise for every $t \in [0, T]$.
The passage to the limit is performed using Skorohod representation theorem.
The main work is to prove that

$$E\left[\left(\left|\int_0^t \left\langle H_\phi, \omega_s^{N_k} \otimes \omega_s^{N_k} \right\rangle ds - \int_0^t \left\langle H_\phi, \omega_s \otimes \omega_s \right\rangle ds\right|\right) \wedge 1\right] \to 0.$$

Here all the detailed informations proved above are used.

Ongoing research and perspectives

- In a forthcoming work with G. Da Prato and M. Röckner we investigate *directly* the associated **Continuity Equation** and prove existence in LlogL class.
- Extension to a *stochastic* 2D Euler equation with **transport type noise** has been proved in:

F. Flandoli, D. Luo, ρ-white noise solution to 2D stochastic Euler equations, arXiv:1710.04017.

- The stochastic case allows us to prove a **gradient estimate** for solutions to the continuity (Fokker-Planck) equation.
- It identifies a new class of transport type noise, with *singular quadratic variation*, which is not (yet) well defined but could produce *regularization by noise*.

2D Euler equations with stochastic transport term

Consider

$$d\omega + u \cdot \nabla\omega dt + \nabla\omega \circ dW = 0$$
$$\operatorname{div} u = 0, \qquad \nabla^{\perp} u = \omega$$

which originates by the substitution

$$u \longrightarrow u + \frac{\partial W}{\partial t}.$$

Here

$$W(t,x) := \sum_{k \in \mathbb{Z}^2 \setminus \{0\}} \sigma_k(x) W_t^k$$

$$\sigma_k(x) \sim |k|^{-\alpha} e_k(x) \qquad e_k(x) \sim \frac{k^{\perp}}{|k|} e^{ik\cdot x}.$$

The exponent α corresponds to space-regularity of the noise.

Transport noise regularizes point vortices (2014)

When $\omega_0 = \sum_{i=1}^{N} \xi_i \delta_{X_0^i}$, the solution $\omega_0 = \sum_{i=1}^{N} \xi_i \delta_{X_t^i}$ of the SPDE above corresponds to the dynamics

$$dX_t^i = \sum_{j=1}^{N} \xi_j K\left(X_t^i - X_t^j\right) dt + \sum_k \sigma_k\left(X_t^i\right) dW_t^k.$$

Theorem (F.-Gubinelli-Priola SPA 2014)

There exist σ_k such that **for every** *initial condition $\left(X_0^{1,N}, \ldots, X_0^{N,N}\right) \in \Delta_N^c$, there is a unique solution $\left(X_t^{1,N}, \ldots, X_t^{N,N}\right)$ with the property that $\left(X_t^{1,N}, \ldots, X_t^{N,N}\right) \in \Delta_N^c$ for all $t \geq 0$.*

Transport noise leads to gradient estimates (2017)

- In the paper F. Flandoli, D. Luo, arXiv:1710.04017 we have proved that the density $\rho_s(\omega)$ of the constructed solution to Euler equations satisfies the gradient estimate

$$\int_0^t \int \left(\sum_k \langle \sigma_k \cdot \nabla \omega, \nabla_{L^2} \rho_s(\omega) \rangle_{L^2}^2 \right) \mu(d\omega)\, ds \leq C.$$

- In principle, such estimate could be the key to prove a uniqueness result (regularization by noise).
- However, it is a degenerate estimate, $\sigma_k \cdot \nabla \omega = 0$ at $\omega = 0$.
- The difficult problem is to estimate the (unbounded) drift by this gradient term.

Singular noise (future research)

- We succeed to control the (unbounded) drift by this gradient term only when

$$\sigma_k(x) \sim |k|^{-1} e_k(x) \qquad e_k(x) \sim \frac{k^\perp}{|k|} e^{ik\cdot x}.$$

- However, in this case, Itô-Stratonovich corrector, and quadratic variation, blow-up logaritmically.
- The previous existence theory holds only for $\sigma_k \sim |k|^{-\alpha} e_k$, $\alpha > 1$.
- Preliminary work on Shnirelmann counter-example, also indicate that $\sigma_k \sim |k|^{-1} e_k$ should be the right noise for regularization.
- Understanding of this singular case is open.

Summary of results and questions

1. We have proved Albeverio-Cruzeiro (CMP '90) result using a classical PDE approach called *weak vorticity formulation*, plus some white noise analysis
2. We have proved that Albeverio-Cruzeiro is the *limit of random point vortices*
3. and that they are the *limit of* L^∞ *solutions*
4. We have extended Albeverio-Cruzeiro result to some class of *absolutely continuous initial conditions*
5. and to a stochastic case, with *transport noise*
6. Similarly to *regularization by noise* for point vortices, the effect of transport noise on Albeverio-Cruzeiro theory is under investigation.

Thank you for your attention