STOCHASTIC MODELS FOR WIND ENERGY MARKETS

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Abstract

Wind energy is a fast growing source of energy. Its power is strictly linked to the wind dynamics, which is very hard to predict. To solve latter issue, we have successfully proposed a stochastic approach. Using different stochastic techniques, we derive an analytical formula for the income from a wind mill as the product of the electricity Spot price and the amount of energy produced by the implant. Calibration problems are also addressed, as well as a treatment of associated quanto options.

Spot price model

Let the Spot price process S be modelled by

$$S(t) = \Lambda_S(t)e^{K(t)},\ t \geq t_0 \geq 0, \tag{1}$$

where

- Λ_S is a deterministic process for the seasonal behaviour,
- K is the stochastic component driven by

$$dK(t) = -\alpha_S K(t)dt + \sigma_S dB_S(t) + dL(t), \tag{2}$$

for

- α_S, $\sigma_S > 0$ constant parameters,
- B_S Brownian motion,
- L Lévy process with characteristic triplet (A, ν, γ), independent from B_S.

Introducing the two following processes

$$Y(t) := e^{-\alpha_S(t-t_0)}\left\{K(t_0) + \sigma_S\int_{t_0}^t e^{\alpha_S(s-t_0)}dB_S(s)\right\},$$

$$Z(t) := e^{-\alpha_S(t-t_0)}\int_{t_0}^t e^{\alpha_S(s-t_0)}dL(s),$$

we can rewrite K as a sum, namely

$$K(t) = Y(t) + Z(t), \qquad t \geq t_0 \geq 0, \tag{3}$$

where, in particular, Z and Y are independent processes, and Y is nothing but an Ornstein-Uhlenbeck process with initial value $Y(t_0) = K(t_0)$.

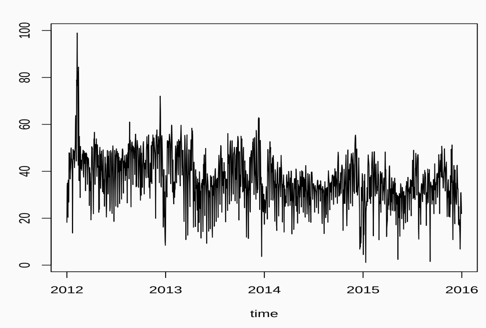

Fig. 1: Spot price time series.

Modelling the jumps

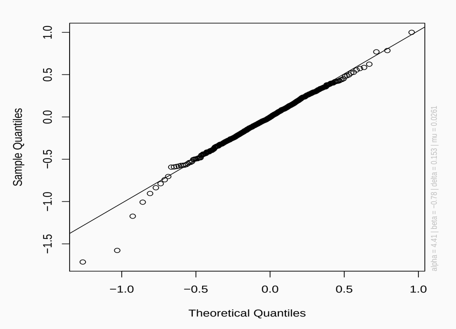

Fig. 2: QQ plot for the NIG distribution.

Exploiting results on NIG processes applied to spot prices dynamics for oil and natural gas, in Figure 2 we can see how the theoretical NIG quantiles fit quite well the sample ones.

Approximation result
A NIG Lévy process X with Lévy measure ν can be approximated by the sum of a scaled Brownian motion B and a Compound Poisson process C, namely

$$X_\epsilon(t) := \sigma_\epsilon B(t) + C(t),\ t \geq 0.$$

Where

- $\sigma_\epsilon^2 := \int_{|x|<\epsilon} x^2\nu(dx)$ is the variation of the small jumps of X,
- C is independent from B, with Lévy measure given by $\nu_\epsilon(dx) := \nu(dx)\mathbb{I}_{|x|\geq\epsilon}$.

Introducing the NIG Lévy process $\tilde{L}$ with parameters $(\alpha_N, \beta_N, \mu_N, \delta_N)$ as the sum given by $\tilde{L}(t) := \sigma_S B_S(t) + L(t)$, $t \geq t_0 \geq 0$, eq. (2) becomes

$$dK(t) = -\alpha_S K(t)dt + d\tilde{L}(t).$$

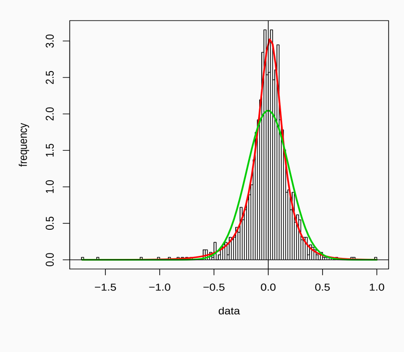

Fig. 3: The estimated NIG distribution.

Figure 3 shows how the bell-shaped curve of the estimated NIG distribution (red line) fits the sample data better than the estimated Gaussian distribution (green line).

	Estimate	CI (95%)		Estimate	CI (95%)
$\hat{\alpha}_S$	1.232	$[1.077, 1.416]$	$\hat{\mu}_N$	0.0261	$[0.0124, 0.0399]$
$\hat{\alpha}_N$	4.413	$[3.493, 5.333]$	$\hat{\delta}_N$	0.153	$[0.133, 0.173]$
$\hat{\beta}_N$	-0.780	$[-1.282, -0.277]$	$\hat{\sigma}_S$	$9.87 \cdot 10^{-4}$	$[9.13, 10.5] \cdot 10^{-4}$

Fig. 4: Estimated arameters.

While in Table 4 we reported the estimated values for the parameters together with 95% confidence intervals.

Calibration and pricing

Calibrations and pricing procedures have been conducted considering daily data from 1/1/2012 to 31/12/2015, for a total of 1460 samples.

- Spot price: Phelix Day Base (PDB) index from the European Power Exchange (EPEX) Spot Daily indices,
- Wind speed: data sets from Karlsruhe Institute of Technology (KIT), measured at 100 metres above the ground,
- Power production: data retrieved from www.50hertz.com.

The computations have been conducted using the software R.

Models for wind power production

Betz's law
In 1919 the German physicist Albert Betz stated that the power production from a wind mill is proportional to the cubic wind speed:

$$P(t) = h\,W^3(t)\,\mathbb{I}_{m\leq W(t)\leq M}.$$

Where

- h is the heat rate, a real constant measuring the efficiency of the wind mill,
- $m \in \mathbb{R}^+$ is the cut-in wind speed, the minimum wind speed to produce energy,
- $M \in \mathbb{R}^+$ is the cut-off wind speed, the maximum wind speed to produce energy.

Wind speed
We model the wind speed W by

$$W(t) = \Lambda_W(t)e^{X(t)},\ t \geq t_0 \geq 0, \tag{4}$$

where

- Λ_W is a deterministic process for the seasonal behaviour,
- X is the stochastic component driven by

$$dX(t) = -\alpha_W X(t)dt + \sigma_W dB_W(t), \tag{5}$$

for

- α_W, $\sigma_W > 0$ constant parameters,
- B_S Brownian motion dependent from B_S in eq. (2) with constant correlation $\rho \in (-1, 1)$, but independent from L.

	Estimate	CI (95%)		Estimate	CI (95%)
$\hat{\alpha}_W$	0.797	$[0.700, 0.904]$	$\hat{\sigma}_W^2$	0.115	$[0.107, 0.124]$

Fig. 5: Estimated parameters.

In Table 5 we reported the estimated values for the parameters together with 95% confidence intervals.

Power production
We model the power production P from a wind mill by

$$P(t) = \Lambda_P(t)e^{Q(t)},\ t \geq t_0 \geq 0,$$

where

- Λ_P is a deterministic process for the seasonal behaviour,
- Q is the stochastic component given by $Q(t) := \mathbf{b}^T\mathbf{R}(t)$, with $\mathbf{R}$ driven by

$$d\mathbf{R}(t) = A\mathbf{R}(t)dt + \mathbf{e}_p\sigma_W dB_W(t),\ t \geq t_0 \geq 0,$$

for

- $\mathbf{e}_p$ the p-th Euclidean basis vector of $\mathbb{R}^p$,
- $\sigma_W > 0$ constant parameter,
- B_W the same Brownian motion as for W,
- $A \in \mathbb{R}^{p\times p}$ an invertible matrix,
- $0 \leq q < p$, $\mathbf{b} = [b_0, ..., b_{p-1}]$, such that $b_q = 1$ and $b_j = 0$ for $q < j < p$.

In particular we set

- $q = 0$, in order to simplify the calibration,
- $p = 3$, exploiting the analysis of the auto-correlation function.

	Estimate	CI (95%)
$\hat{\alpha}_1$	2.36	$[2.31, 2.41]$
$\hat{\alpha}_2$	1.90	$[1.73, 2.07]$
$\hat{\alpha}_3$	0.478	$[0.315, 0.640]$
$\hat{\lambda}_1$	$-0.482 + 0i$	$[-0.488, -0.476] \times [-0.006, 0.006]$
$\hat{\lambda}_2$	$-0.940 - 0.327i$	$[-0.947, -0.933] \times [-0.333, -0.321]$
$\hat{\lambda}_3$	$-0.940 + 0.327i$	$[-0.946, -0.935] \times [0.315, 0.339]$
$\hat{\sigma}_W^2$	0.283	$[0.264, 0.305]$

Fig. 6: Estimated parameters.

In Table 6 we reported the estimated values for the parameters together with 95% confidence intervals.

Income from a Wind Power Plant

Given $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{Q})$, the income at time $t_0 \geq 0$ from a wind mill up to time $t \geq t_0$ is given by

Income formula

$$V_0 := \mathbb{E}_\mathbb{Q}\left[\int_{t_0}^t e^{-r(s-t_0)}P(s)S(s)ds \,\Big|\, \mathcal{F}_{t_0}\right],$$

hence, by the Betz's Law, we have $V_0 = h\int_{t_0}^t e^{-r(s-t_0)}f(s)ds$, for

$$f(t) := \mathbb{E}_\mathbb{Q}\left[W^3(t)\mathbb{I}_{\{m\leq W(t)\leq M\}}S(t)\,\Big|\,\mathcal{F}_{t_0}\right].$$

By equations (1), (3), (4), and (5), we can then express f by the product $f(t) = f_{XY}(t)f_Z(t)$, where

$$f_{XY}(t) := \Lambda_W^3(t)\Lambda_S(t)\,\mathbb{E}_\mathbb{Q}\left[e^{3X(t)+Y(t)}\mathbb{I}_{\{A\leq X(t)\leq B\}}\,\Big|\,\mathcal{F}_{t_0}\right], \tag{6}$$

$$f_Z(t) := \mathbb{E}_\mathbb{Q}\left[e^{Z(t)}\,\Big|\,\mathcal{F}_{t_0}\right], \tag{7}$$

which implies:

Theorem 1. *The analytical expression for function (6) is given by*

Price formula - Brownian component

$$f_{XY}(t) = \Lambda_W^3(t)\Lambda_S(t)\Big[\Phi(d(t,M)) - \Phi(d(t,m))\Big]\exp\left\{\frac{1}{2}\Sigma^2(t) + (3\mu_X(t) + \mu_Y(t))\right\},$$

where Φ is the cumulative standard Gaussian distribution function, while

$$\Sigma^2(t) := 9\sigma_X^2(t) + 6c(t)\sigma_X^2(t) + \sigma_Y^2(t),$$

and, indicating with k the variable which can be equal to m or to M, $d(t,k)$ is defined by

$$d(t,k) := \frac{\ln\left(\frac{k}{\Lambda_W(t)}\right) - \mu_X(t)}{\sigma_X(t)} - (3 + c(t))\sigma_X(t),$$

for $c(t) := \mathrm{Corr}(X(t), Y(t))$.

Proof. Taking the expectation in eq. (6), we are left with a double integral, where (X, Y) is nothing but a 2-dimensional Gaussian random vector. □

Theorem 2. *The analytical expression for function (7) is given by*

Price formula - Lévy component

$$f_Z(t) = \exp\left\{\int_{t_0}^t\int_{-\infty}^\infty\left(e^{\alpha_S(s-t)}x - 1\right)\nu(dx)ds\right\},$$

where $\nu(dx)$ is the Lévy measure for the Compound Poisson process L.

Proof. Ask the expert... here I am! □

Quanto options

We consider an **European put-type quanto option** to cover against:

- price risks:

Spot price index

$$E := \frac{1}{\tau_2 - \tau_1}\sum_{t=\tau_1}^{\tau_2} S(t),$$

- risk due to the plants' wear or weak production:

Power production index

$$I := \sum_{t=\tau_1}^{\tau_2}\left(hW^3(t) - P(t)\right).$$

If the option is exercised at time τ_2, its arbitrage-free price at time $t_0 \leq \tau_2$ is given by

$$C_0 := e^{-r(\tau_2-t_0)}\mathbb{E}_\mathbb{Q}\left[f(E, I)\,\Big|\,\mathcal{F}_{t_0}\right], \tag{8}$$

where:

- $r > 0$ is the risk-free interest rate, • $\mathbb{Q}$ is the risk-neutral measure,
- $\mathcal{F}_t$ represents the filtration, • $[\tau_1, \tau_2]$ is the measurement period,
- $t \leq \tau_2$ being the evaluation time, • f is the payoff function.

We relate the price of the quanto option in eq. (8) to futures contracts on the energy and wind production indexes: the price at time $t_0 \leq \tau_2$ of a futures contract written on S with delivery period $[\tau_1, \tau_2]$ is given by

$$F_{t_0}^E(\tau_1, \tau_2) = \mathbb{E}_\mathbb{Q}\left[\frac{1}{\tau_2 - \tau_1}\sum_{t=\tau_1}^{\tau_2} S(t)\,\Big|\,\mathcal{F}_{t_0}\right],$$

and $F_{\tau_2}^E(\tau_1, \tau_2) = E$; similarly

$$F_{t_0}^I(\tau_1, \tau_2) = \mathbb{E}_\mathbb{Q}\left[\sum_{t=\tau_1}^{\tau_2}\left(hW^3(t) - P(t)\right)\,\Big|\,\mathcal{F}_{t_0}\right],$$

and $F_{\tau_2}^I(\tau_1, \tau_2) = I$.
Therefore, we are considering the above defined quanto option as written on two futures contracts:

Option price

$$C_0 = e^{-r(\tau_2-t_0)}\mathbb{E}_\mathbb{Q}\left[\max\left(K_E - F_{\tau_2}^E(\tau_1,\tau_2), 0\right)\cdot\max\left(K_I - F_{\tau_2}^I(\tau_1,\tau_2), 0\right)\,\Big|\,\mathcal{F}_t\right].$$

Motivated by pricing purposes, we introduce the following dynamics:

- **Price index dynamics:** $F_T^E(\tau_1, \tau_2) := F_t^E(\tau_1, \tau_2)\exp(\mu_E + Z_E)$,
- **Power index dynamics:** $F_T^I(\tau_1, \tau_2) := F_t^I(\tau_1, \tau_2)\exp(\mu_I + Z_I)$,

where

- Z_E and Z_I two $\mathcal{F}_t-$measurable Gaussian random variables with correlation ρ,
- $\mu_{Z_E} = 0$, $\mu_{Z_I} = 0$, $\sigma_{Z_E}^2 := \mathrm{Var}(Z_E)$, $\sigma_{Z_I}^2 := \mathrm{Var}(Z_I)$,
- $\mu_E = -\frac{\sigma_{Z_E}^2}{2}$ and $\mu_I = -\frac{\sigma_{Z_I}^2}{2}$, to guarantee the martingale property,

so that, we have:

Theorem 3. *The market price at time t_0 of the European energy quanto option exercised at time τ_2 is given by*

Price function

$$C_0 = e^{-r(\tau_2-t_0)}\Big(K_EK_IM(x_1, y_1;\rho) - K_IF_{t_0}^E(\tau_1,\tau_2)M(x_2, y_2;\rho) + -K_EF_{t_0}^I(\tau_1,\tau_2)M(x_3, y_3;\rho) + F_{t_0}^E(\tau_1,\tau_2)F_{t_0}^I(\tau_1,\tau_2)e^{\rho\sigma_{Z_E}\sigma_{Z_I}}M(x_4, y_4;\rho)\Big),$$

with

$$x_1 := \frac{\log\left(\frac{K_I}{F_{t_0}^I(\tau_2,\tau_1)}\right) + \frac{\sigma_{Z_I}^2}{2}}{\sigma_{Z_I}}, \qquad y_1 := \frac{\log\left(\frac{K_E}{F_{t_0}^E(\tau_2,\tau_1)}\right) + \frac{\sigma_{Z_E}^2}{2}}{\sigma_{Z_E}},$$

$$x_2 := x_1 - \rho\sigma_{Z_I}, \quad x_3 := x_1 - \sigma_{Z_E}, \quad x_4 := x_1 - \sigma_{Z_E} - \rho\sigma_{Z_I},$$

$$y_2 := y_1 - \sigma_{Z_I}, \quad y_3 := y_1 - \rho\sigma_{Z_E}, \quad y_4 := y_1 - \sigma_{Z_I} - \rho\sigma_{Z_E},$$

where $M(x, y;\rho)$ denotes the standard bivariate Gaussian cumulative distribution function with correlation ρ.

Proof. See Proposition 3.1 in [4]. □

ρ	K_I \ K_E	Three-years 100	Three-years 150	Three-years 200	Two-years 100	Two-years 150	Two-years 200	One-year 100	One-year 150	One-year 200
-0.95	$1.0 \cdot 10^6$	8.9259	46.939	84.950	31.482	69.897	108.31	57.369	96.467	135.57
	$1.5 \cdot 10^6$	14.752	77.517	140.28	51.803	114.97	178.14	93.713	157.56	221.41
	$2.0 \cdot 10^6$	20.579	108.10	195.61	72.124	160.04	247.96	130.06	218.66	307.26
-0.50	$1.0 \cdot 10^6$	8.9367	46.949	84.961	31.509	69.925	108.34	57.389	96.487	135.59
	$1.5 \cdot 10^6$	14.763	77.528	140.29	51.830	115.00	178.16	93.733	157.58	221.43
	$2.0 \cdot 10^6$	20.590	108.11	195.62	72.152	160.07	247.99	130.08	218.68	307.28
0	$1.0 \cdot 10^6$	8.9487	46.961	84.972	31.540	69.956	108.37	57.411	96.509	135.61
	$1.5 \cdot 10^6$	14.776	77.540	140.30	51.862	115.03	178.20	93.755	157.60	221.45
	$2.0 \cdot 10^6$	20.603	108.12	195.63	72.183	160.10	248.02	130.10	218.70	307.31
0.50	$1.0 \cdot 10^6$	8.9608	46.973	84.984	31.571	69.987	108.40	57.433	96.532	135.63
	$1.5 \cdot 10^6$	14.788	77.551	140.31	51.893	115.06	178.23	93.777	157.63	221.48
	$2.0 \cdot 10^6$	20.615	108.13	195.64	72.214	160.13	248.05	130.12	218.72	307.32
0.95	$1.0 \cdot 10^6$	8.9717	46.983	84.995	31.599	70.015	108.43	57.453	96.552	135.65
	$1.5 \cdot 10^6$	14.799	77.562	140.32	51.921	115.09	178.25	93.797	157.65	221.50
	$2.0 \cdot 10^6$	20.626	108.14	195.65	72.242	160.16	248.08	130.14	218.74	307.34

Fig. 7: Quanto option prices.

In Table 7 we report the price value obtained considering different values for the strikes, K_I and K_E, different values of the correlation ρ and various calibration periods. All values must be multiplied by 10^6.

	Three-years	Two-years	One-year
Lévy-driven Spot price	-0.949	0.712	0.660
Ornstein-Uhlenbeck Spot price	0.531	0.695	0.664

Fig. 8: Correlation.

In Table 8 we report the correlation values obtained.

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